ME303 Introduction to Mechanical Design

Lecture 04
Deflection & Stiffness

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Agenda

Week 03, Wednesday, Sep 18, 2019

- Basic Concept
  - Spring Rates
  - Tension, Compression, and Torsion
- Deflection Due to Bending
  - Beam Deflection Methods
  - Beam Deflections by Superposition
  - Appendix Tables
  - Examples
- Strain Energy
  - Castigliano's Theorem
- Statically Indeterminate Problems
  - Procedures 1 & 2
  - Examples
- Stability & Instability
  - Global Instability – Buckling
  - Elastic Stability
- Shock & Impact

Student Team Selection

https://jinshuju.net/f/79xHV7

Please fill out this form to register your team.
Only the team leader is required to fill out this form.
All team members are recommended to be within the same group.
Each team can only have 5~6 students, no more, no less.

The following students have volunteered to be the Team Leaders.

- Group 1-Team 1-Team Leader:
  Zhang Zicong (11711203@mail.sustech.edu.cn)
- Group 1-Team 2-Team Leader:
  Yao Shilong (11711721@mail.sustech.edu.cn)
- Group 1-Team 3-Team Leader:
  [Opening Position for students in Group 1]
- Group 1-Team 4-Team Leader:
  [Opening Position for students in Group 1]

- Group 2-Team 5-Team Leader:
  Sui Xiaodi (11712839@mail.sustech.edu.cn)
- Group 2-Team 6-Team Leader:
  Wang Haowen (11711020@mail.sustech.edu.cn)
- Group 2-Team 7-Team Leader:
  Chen Xuanwu (11710323@mail.sustech.edu.cn)
- Group 2-Team 8-Team Leader:
  Qiao Jixiang (11710817@mail.sustech.edu.cn)

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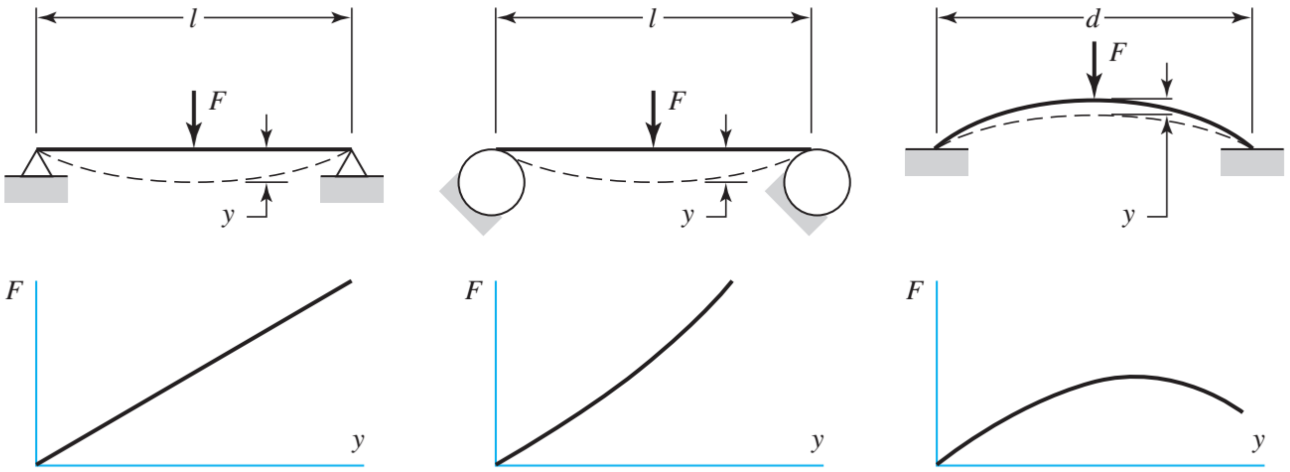
Spring Rates

Elastic Modeling of Any Body



(a) A linear spring; (b) a stiffening spring; (c) a softening spring.

$$F = F(y)$$

$$k(y) = \lim_{\Delta y \to 0} \frac{\Delta F}{\Delta y} = \frac{dF}{dy}$$

$$k = \frac{F}{y}$$

Spring Constant

- Elasticity
    - *The ability to regain its original configuration after having been deformed.*
- A ***spring*** is a mechanical element that exerts a force when deformed.
    - Also applicable for Torques and Moments

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Tension, Compression, and Torsion

Common Metrics for Further Analysis

The total extension or contraction of a uniform bar in pure tension or compression

$$\delta = \frac{Fl}{AE}$$

$$\downarrow \quad k = \frac{F}{y}$$

$$k = \frac{AE}{l}$$

Not applicable to a long bar due to a possibility of buckling

The angular deflection of a uniform solid or hollow round bar subjected to a twisting moment *T*

$$\theta = \frac{Tl}{GJ}$$

$$\downarrow \quad k = \frac{F}{y}$$

$$k = \frac{T}{\theta} = \frac{GJ}{l}$$

Applicable *only* to circular cross sections

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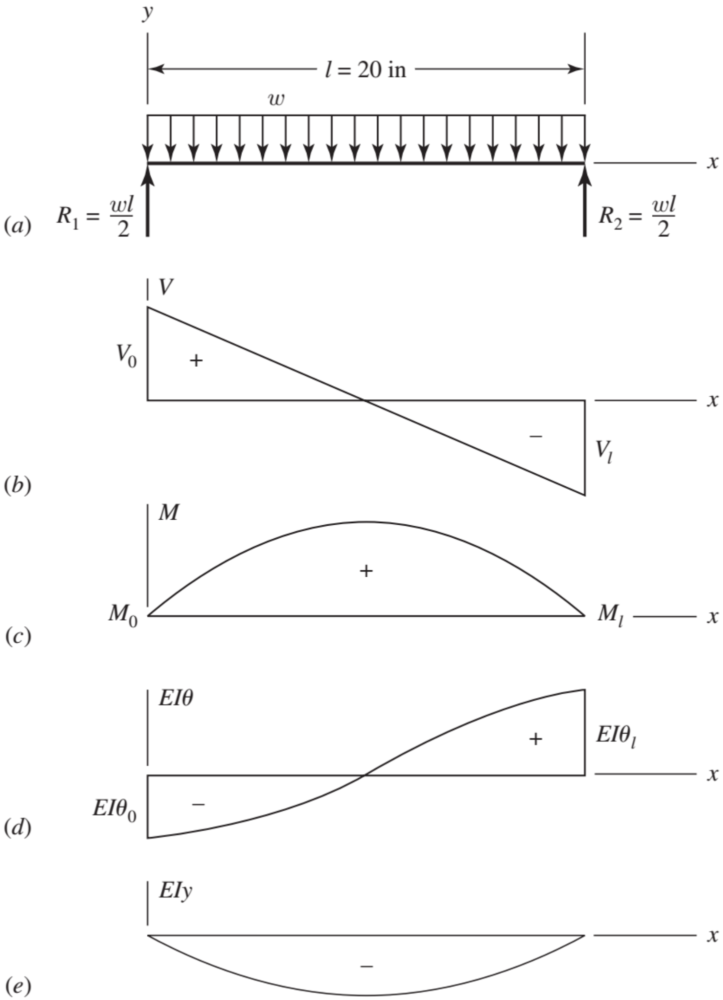
Deflection Due to Bending

The problem of bending of beams probably occurs more often than any other loading problem in mechanical design.



Loading, w
$w = 80$ lbf/in

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

Shear, V
$V_0 = +800$ lbf
$V_l = -800$ lbf

$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

Moment, M
$M_0 = M_l = 0$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

Slope, $EI\theta$
$\theta_{l/2} = 0$

$$\theta = \frac{dy}{dx}$$

Deflection, EIy
$y_0 = y_l = 0$

$$y = f(x)$$

The curvature of a beam subjected to a bending moment M is given by

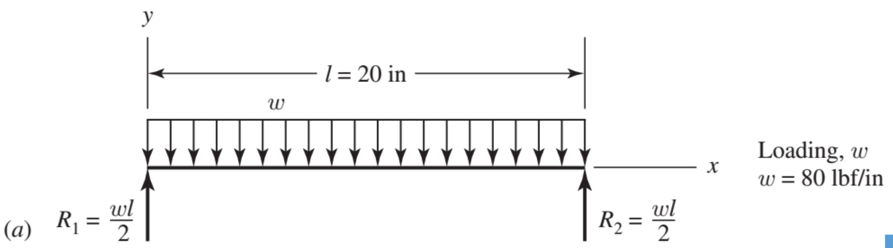
$$\frac{1}{\rho} = \frac{M}{EI} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

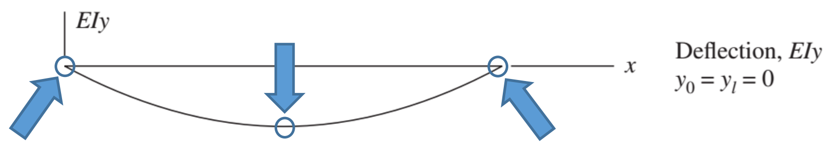
the radius of curvature

For many problems in bending, the slope is *very small*, and for these the denominator can be taken as *unity*.

Determine the Slope and Deflection of the beam

A Simple Example





the bending moment equation, for $0 \le x \le l$, is

$$M = \frac{wl}{2}x - \frac{w}{2}x^2$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$EI\frac{dy}{dx} = \int M\,dx = \frac{wl}{4}x^2 - \frac{w}{6}x^3 + C_1$$

$$EI\,y = \iint M\,dx = \frac{wl}{12}x^3 - \frac{w}{24}x^4 + C_1x + C_2$$

Boundary Conditions $y = 0$ at $x = 0$ and l.

$C_2 = 0,$ $C_1 = -wl^3/24.$

Deflection $$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

Slope $$\theta = \frac{dy}{dx} = \frac{w}{24EI}(6lx^2 - 4x^3 - l^3)$$

$$\theta|_{x=0} = -\frac{wl^3}{24EI}$$ $$\theta|_{x=l} = \frac{wl^3}{24EI}$$

$$dy/dx = 0$$ $$y_{\max} = -\frac{5wl^4}{384EI}$$

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Beam Deflection Methods

For beams with discontinuous loading and/or geometry.

i.e. simple loading for direct solution, **such as those in the Appendix Table-9**

i.e. Fourier series

i.e. calculated estimation

Forms of solution:

1. Closed-form, or
2. Represented by infinite series, which amount to closed form if the series are rapidly convergent, or
3. Approximations obtained by evaluating the first or the first and second terms.

Some popular methods:

- Superposition
- The moment-area method
- Singularity functions
- Numerical integration

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Beam Deflections by Superposition

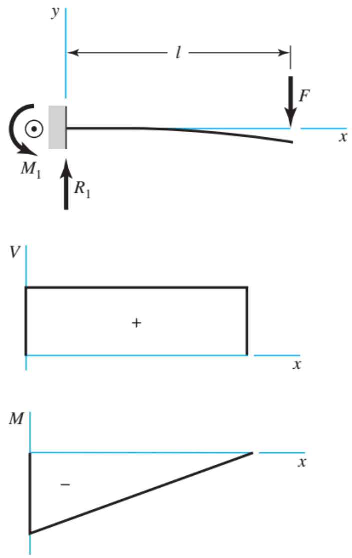
1+1=2

- ***<u>Superposition</u>***
  - resolves the effect of combined loading on a structure by determining the effects of each load separately and adding the results algebraically.

- Applicable Conditions:
  - (1) each effect is linearly related to the load that produces it,
  - (2) a load does not create a condition that affects the result of another load, and
  - (3) the deformations resulting from any specific load are not large enough to appreciably alter the geometric relations of the parts of the structural system.

- ***Make the best use of Tables (Simple, Common Solutions)***

1 Cantilever—end load



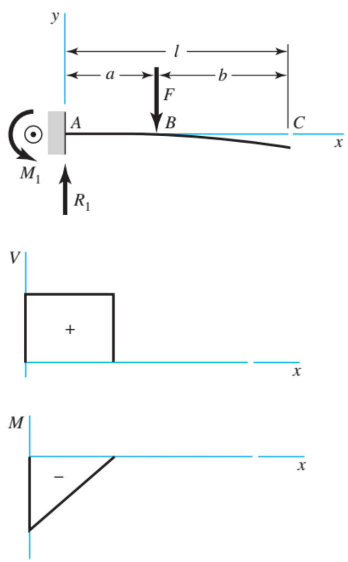
$$R_1 = V = F \qquad M_1 = Fl$$

$$M = F(x - l)$$

$$y = \frac{Fx^2}{6EI}(x - 3l)$$

$$y_{max} = -\frac{Fl^3}{3EI}$$

2 Cantilever—intermediate load



$$R_1 = V = F \qquad M_1 = Fa$$

$$M_{AB} = F(x - a) \qquad M_{BC} = 0$$

$$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$$

$$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$$

$$y_{max} = \frac{Fa^2}{6EI}(a - 3l)$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

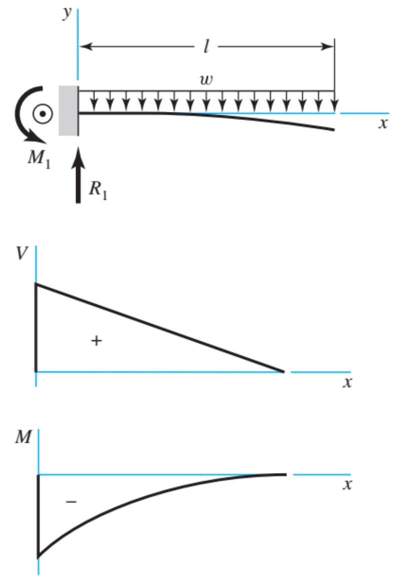
$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\theta = \frac{dy}{dx}$$

$$y = f(x)$$

3 Cantilever—uniform load



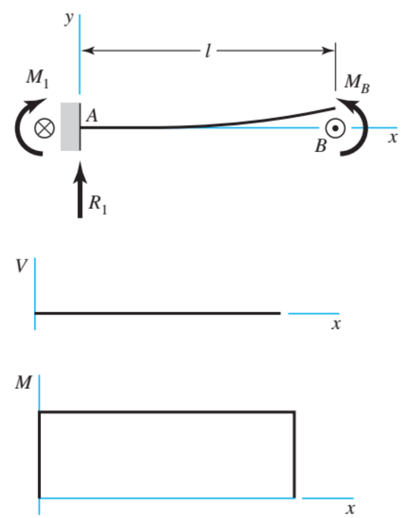
$$R_1 = wl \qquad M_1 = \frac{wl^2}{2}$$

$$V = w(l - x) \qquad M = -\frac{w}{2}(l - x)^2$$

$$y = \frac{wx^2}{24EI}(4lx - x^2 - 6l^2)$$

$$y_{max} = -\frac{wl^4}{8EI}$$

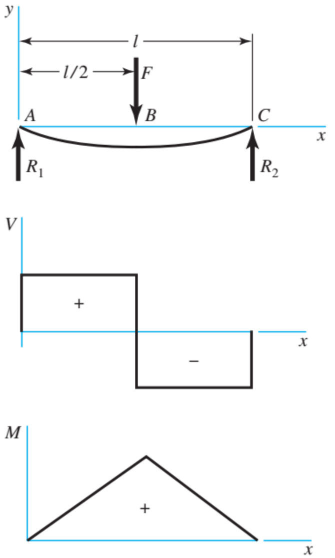
4 Cantilever—moment load



$$R_1 = V = 0 \qquad M_1 = M = M_B$$

$$y = \frac{M_B x^2}{2EI} \qquad y_{max} = \frac{M_B l^2}{2EI}$$

5 Simple supports—center load



$$R_1 = R_2 = \frac{F}{2}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fx}{2} \qquad M_{BC} = \frac{F}{2}(l - x)$$

$$y_{AB} = \frac{Fx}{48EI}(4x^2 - 3l^2)$$

$$y_{max} = -\frac{Fl^3}{48EI}$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

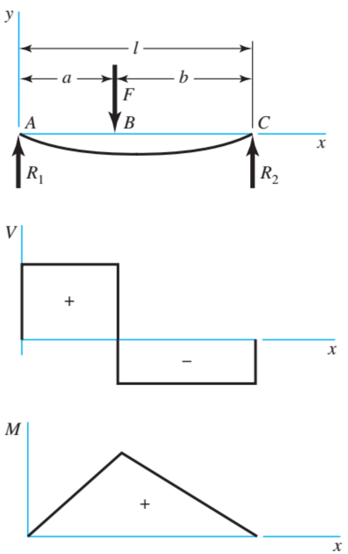
$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\theta = \frac{dy}{dx}$$

$$y = f(x)$$

6 Simple supports—intermediate load



$$R_1 = \frac{Fb}{l} \qquad R_2 = \frac{Fa}{l}$$

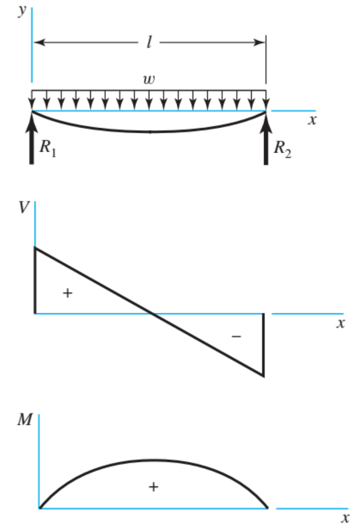
$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \qquad M_{BC} = \frac{Fa}{l}(l - x)$$

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l - x)}{6EIl}(x^2 + a^2 - 2lx)$$

7 Simple supports—uniform load



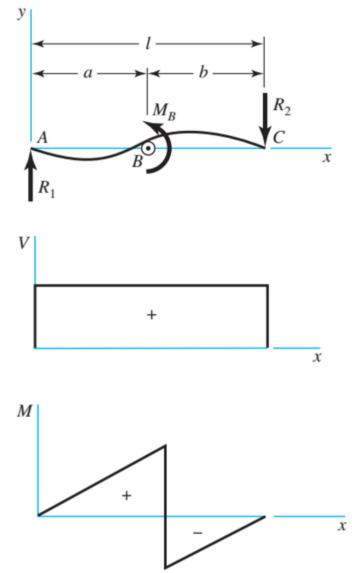
$$R_1 = R_2 = \frac{wl}{2} \qquad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{max} = -\frac{5wl^4}{384EI}$$

8 Simple supports—moment load



$$R_1 = R_2 = \frac{M_B}{l} \qquad V = \frac{M_B}{l}$$

$$M_{AB} = \frac{M_B x}{l} \qquad M_{BC} = \frac{M_B}{l}(x - l)$$

$$y_{AB} = \frac{M_B x}{6EIl}(x^2 + 3a^2 - 6al + 2l^2)$$

$$y_{BC} = \frac{M_B}{6EIl}[x^3 - 3lx^2 + x(2l^2 + 3a^2) - 3a^2l]$$

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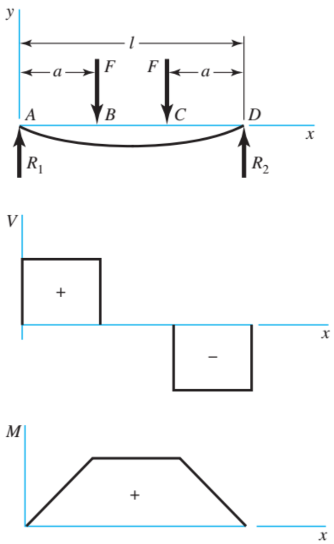
9 Simple supports—twin loads



$$R_1 = R_2 = F \qquad V_{AB} = F \qquad V_{BC} = 0$$

$$V_{CD} = -F$$

$$M_{AB} = Fx \qquad M_{BC} = Fa \qquad M_{CD} = F(l - x)$$

$$y_{AB} = \frac{Fx}{6EI}(x^2 + 3a^2 - 3la)$$

$$y_{BC} = \frac{Fa}{6EI}(3x^2 + a^2 - 3lx)$$

$$y_{max} = \frac{Fa}{24EI}(4a^2 - 3l^2)$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

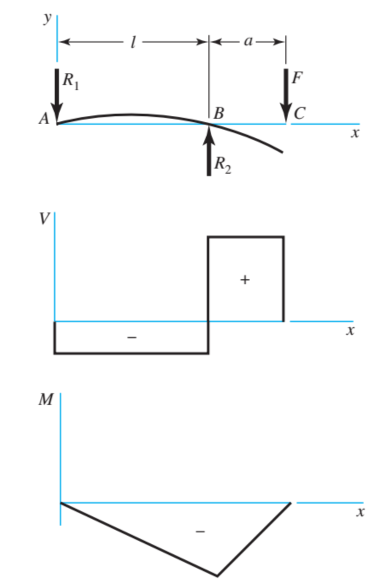
$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\theta = \frac{dy}{dx}$$

$$y = f(x)$$

10 Simple supports—overhanging load



$$R_1 = \frac{Fa}{l} \qquad R_2 = \frac{F}{l}(l + a)$$

$$V_{AB} = -\frac{Fa}{l} \qquad V_{BC} = F$$

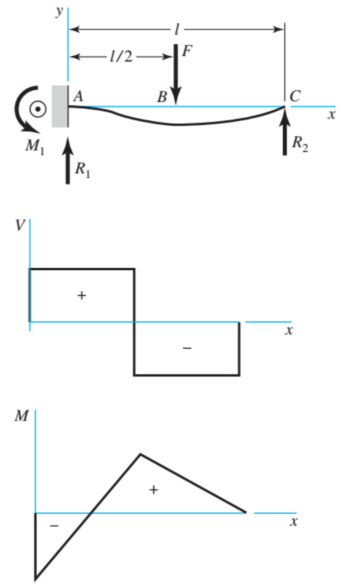
$$M_{AB} = -\frac{Fax}{l} \qquad M_{BC} = F(x - l - a)$$

$$y_{AB} = \frac{Fax}{6EIl}(l^2 - x^2)$$

$$y_{BC} = \frac{F(x - l)}{6EI}[(x - l)^2 - a(3x - l)]$$

$$y_c = -\frac{Fa^2}{3EI}(l + a)$$

11 One fixed and one simple support—center load



$$R_1 = \frac{11F}{16} \qquad R_2 = \frac{5F}{16} \qquad M_1 = \frac{3Fl}{16}$$

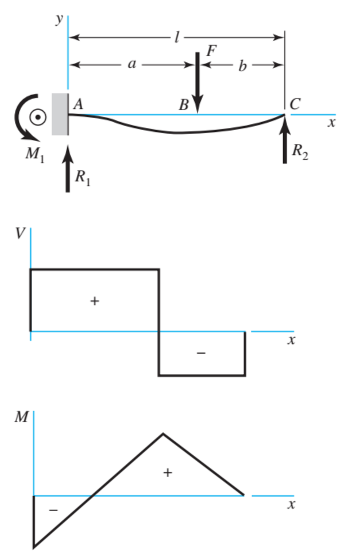
$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{F}{16}(11x - 3l) \qquad M_{BC} = \frac{5F}{16}(l - x)$$

$$y_{AB} = \frac{Fx^2}{96EI}(11x - 9l)$$

$$y_{BC} = \frac{F(l - x)}{96EI}(5x^2 + 2l^2 - 10lx)$$

12 One fixed and one simple support—intermediate load



$$R_1 = \frac{Fb}{2l^3}(3l^2 - b^2) \qquad R_2 = \frac{Fa^2}{2l^3}(3l - a)$$

$$M_1 = \frac{Fb}{2l^2}(l^2 - b^2)$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fb}{2l^3}[b^2l - l^3 + x(3l^2 - b^2)]$$

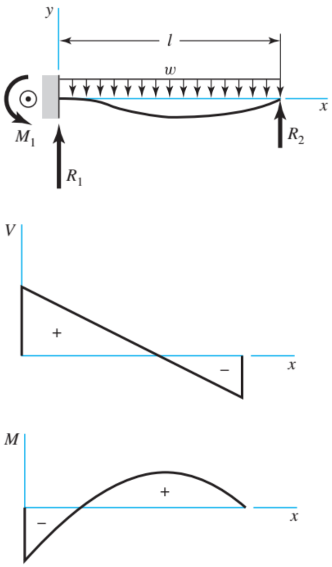
$$M_{BC} = \frac{Fa^2}{2l^3}(3l^2 - 3lx - al + ax)$$

$$y_{AB} = \frac{Fbx^2}{12EIl^3}[3l(b^2 - l^2) + x(3l^2 - b^2)]$$

$$y_{BC} = y_{AB} - \frac{F(x - a)^3}{6EI}$$

(Continued)

13 One fixed and one simple support—uniform load



$$R_1 = \frac{5wl}{8} \qquad R_2 = \frac{3wl}{8} \qquad M_1 = \frac{wl^2}{8}$$

$$V = \frac{5wl}{8} - wx$$

$$M = -\frac{w}{8}(4x^2 - 5lx + l^2)$$

$$y = \frac{wx^2}{48EI}(l - x)(2x - 3l)$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}$$

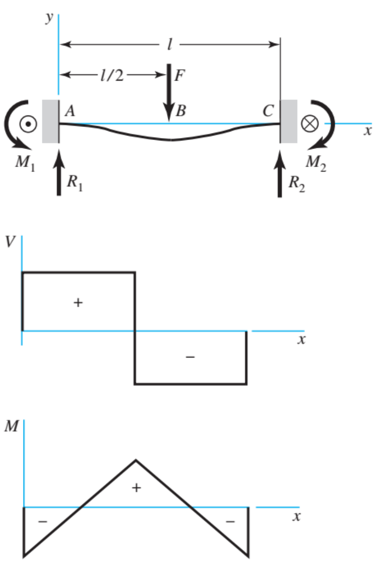
$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\theta = \frac{dy}{dx}$$

$$y = f(x)$$

14 Fixed supports—center load



$$R_1 = R_2 = \frac{F}{2} \qquad M_1 = M_2 = \frac{Fl}{8}$$

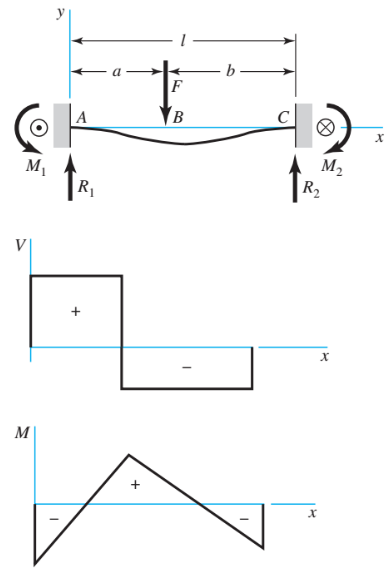
$$V_{AB} = -V_{BC} = \frac{F}{2}$$

$$M_{AB} = \frac{F}{8}(4x - l) \qquad M_{BC} = \frac{F}{8}(3l - 4x)$$

$$y_{AB} = \frac{Fx^2}{48EI}(4x - 3l)$$

$$y_{max} = -\frac{Fl^3}{192EI}$$

15 Fixed supports—intermediate load



$$R_1 = \frac{Fb^2}{l^3}(3a + b) \qquad R_2 = \frac{Fa^2}{l^3}(3b + a)$$

$$M_1 = \frac{Fab^2}{l^2} \qquad M_2 = \frac{Fa^2b}{l^2}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

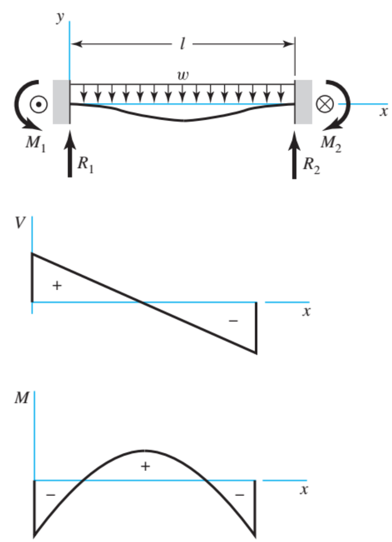
$$M_{AB} = \frac{Fb^2}{l^3}[x(3a + b) - al]$$

$$M_{BC} = M_{AB} - F(x - a)$$

$$y_{AB} = \frac{Fb^2x^2}{6EIl^3}[x(3a + b) - 3al]$$

$$y_{BC} = \frac{Fa^2(l - x)^2}{6EIl^3}[(l - x)(3b + a) - 3bl]$$

16 Fixed supports—uniform load



$$R_1 = R_2 = \frac{wl}{2} \qquad M_1 = M_2 = \frac{wl^2}{12}$$

$$V = \frac{w}{2}(l - 2x)$$

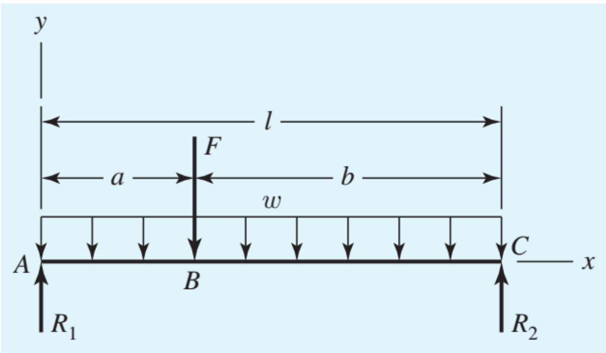
$$M = \frac{w}{12}(6lx - 6x^2 - l^2)$$

$$y = -\frac{wx^2}{24EI}(l - x)^2$$

$$y_{max} = -\frac{wl^4}{384EI}$$

Example 1: Consider a uniformly loaded beam with a concentrated force

Using superposition, determine the reactions and the deflection as a function of *x*.



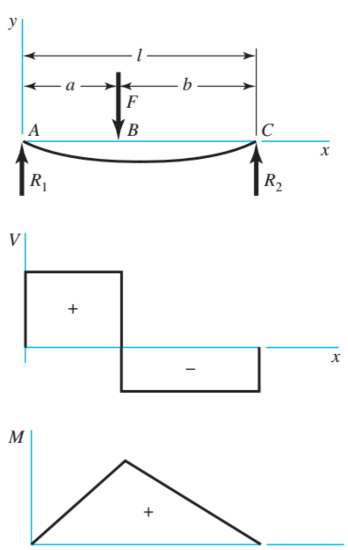
$$R_1 = \frac{Fb}{l} + \frac{wl}{2}$$

$$R_2 = \frac{Fa}{l} + \frac{wl}{2}$$

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2) + \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{BC} = \frac{Fa(l-x)}{6EIl}(x^2 + a^2 - 2lx) + \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

6 Simple supports—intermediate load



$R_1 = \frac{Fb}{l} \qquad R_2 = \frac{Fa}{l}$

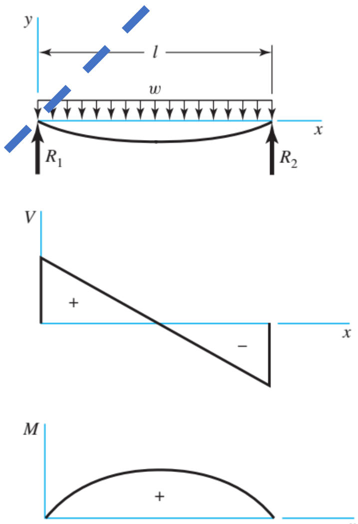
$V_{AB} = R_1 \qquad V_{BC} = -R_2$

$M_{AB} = \frac{Fbx}{l} \qquad M_{BC} = \frac{Fa}{l}(l-x)$

$$y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l-x)}{6EIl}(x^2 + a^2 - 2lx)$$

7 Simple supports—uniform load



$R_1 = R_2 = \frac{wl}{2} \qquad V = \frac{wl}{2} - wx$

$M = \frac{wx}{2}(l-x)$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{max} = -\frac{5wl^4}{384EI}$$

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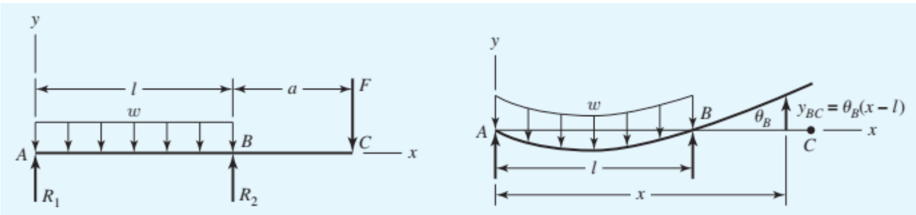
Example 2: beam with uniformly distributed force and overhang force

determine the deflection equations using superposition.



For region AB

$$y_{AB} = \frac{wx}{24EI}(2lx^2 - x^3 - l^3) + \frac{Fax}{6EIl}(l^2 - x^2)$$

For region BC,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{wx}{24EI}(2lx^2 - x^3 - l^3)\right] = \frac{w}{24EI}(6lx^2 - 4x^3 - l^3)$$

$x = l$.

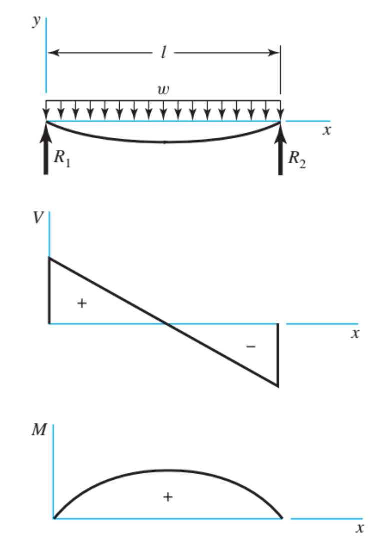
$$\theta_B = \frac{w}{24EI}(6ll^2 - 4l^3 - l^3) = \frac{wl^3}{24EI}$$

the deflection due to F, in BC,

$$y_{BC} = \frac{wl^3}{24EI}(x - l) + \frac{F(x - l)}{6EI}[(x - l)^2 - a(3x - l)]$$

The deflection in region BC due to w is $\theta_B(x - l)$,

7 Simple supports—uniform load



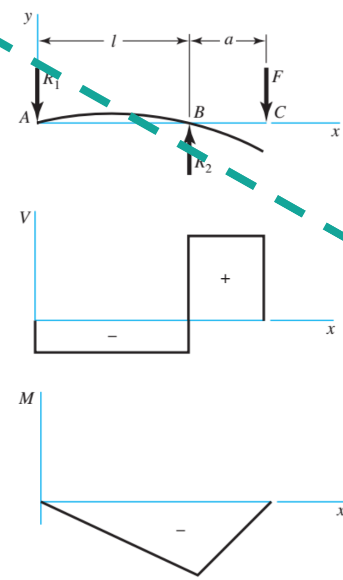
$$R_1 = R_2 = \frac{wl}{2} \qquad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{\max} = -\frac{5wl^4}{384EI}$$

10 Simple supports—overhanging load



$$R_1 = \frac{Fa}{l} \qquad R_2 = \frac{F}{l}(l + a)$$

$$V_{AB} = -\frac{Fa}{l} \qquad V_{BC} = F$$

$$M_{AB} = -\frac{Fax}{l} \qquad M_{BC} = F(x - l - a)$$

$$y_{AB} = \frac{Fax}{6EIl}(l^2 - x^2)$$

$$y_{BC} = \frac{F(x - l)}{6EI}[(x - l)^2 - a(3x - l)]$$

$$y_c = -\frac{Fa^2}{3EI}(l + a)$$

(Continue...

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Strain Energy

The external work done on an elastic member in deforming it is transformed into strain, or potential, energy.

the product of the average force and the deflection

$$U = \frac{F}{2} y = \frac{F^2}{2k}$$

By substituting appropriate expressions for *k*, strain-energy formulas for various simple loadings may be obtained.

$$U = \frac{F^2 l}{2AE}, \quad U = \int \frac{F^2}{2AE} dx \quad \text{tension and compression} \qquad k = \frac{AE}{l}$$

$$U = \frac{T^2 l}{2GJ}, \quad U = \int \frac{T^2}{2GJ} dx \quad \text{torsion} \qquad k = \frac{T}{\theta} = \frac{GJ}{l}$$

$$U = \frac{F^2 l}{2AG}, \quad U = \int \frac{F^2}{2AG} dx \quad \text{direct shear}$$

$$U = \frac{M^2 l}{2EI}, \quad U = \int \frac{M^2}{2EI} dx \quad \text{bending}$$

$$U = \frac{CV^2 l}{2AG}, \quad U = \int \frac{CV^2}{2AG} dx \quad \text{transverse shear}$$

Strain-Energy Correction Factors for Transverse Shear

Beam Cross-Sectional Shape	Factor C
Rectangular	1.2
Circular	1.11
Thin-walled tubular, round	2.00
Box sections[†]	1.00
Structural sections[†]	1.00

[†]Use area of web only.

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Castigliano's Theorem

Virtual Work

- *When forces act on elastic systems subject to small displacements, the displacement corresponding to any force, in the direction of the force, is equal to the partial derivative of the total strain energy with respect to that force.*

$$\delta_i = \frac{\partial U}{\partial F_i}$$

δ_i is the displacement of the point of application of the force F_i in the direction of F_i.

$$\delta = \frac{\partial}{\partial F}\left(\frac{F^2 l}{2AE}\right) = \frac{Fl}{AE}$$

$$\theta_i = \frac{\partial U}{\partial M_i}$$

where θ_i is the rotational displacement, in radians, of the beam where the moment M_i exists and in the direction of M_i.

$$\theta = \frac{\partial}{\partial T}\left(\frac{T^2 l}{2GJ}\right) = \frac{Tl}{GJ}$$

- Castigliano's theorem can be used to find the deflection at a point even though no force or moment acts there.

1. Set up the equation for the total strain energy U by including the energy due to a fictitious force or moment Q acting at the point whose deflection is to be found.
2. Find an expression for the desired deflection δ, in the direction of Q, by taking the derivative of the total strain energy with respect to Q.
3. Since Q is a fictitious force, solve the expression obtained in step 2 by setting Q equal to zero. Thus, the displacement at the point of application of the fictitious force Q is

$$\delta = \left.\frac{\partial U}{\partial Q}\right|_{Q=0}$$

$$\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{AE}\left(F\frac{\partial F}{\partial F_i}\right)dx \qquad \text{tension and compression}$$

$$\theta_i = \frac{\partial U}{\partial M_i} = \int \frac{1}{GJ}\left(T\frac{\partial T}{\partial M_i}\right)dx \qquad \text{torsion}$$

$$\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right)dx \qquad \text{bending}$$

$$\delta_i = \frac{\partial U}{\partial F_i} = \frac{\partial}{\partial F_i}\left(\int \frac{M^2}{2EI}dx\right) = \int \frac{\partial}{\partial F_i}\left(\frac{M^2}{2EI}\right)dx = \int \frac{2M\frac{\partial M}{\partial F_i}}{2EI}dx = \int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right)dx$$

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Statically Indeterminate Problems

Overconstrained with more unknown support (reaction) forces and/ or moments than static equilibrium equations.

Only one equation of static equilibrium can be written, but there are two unknowns.

$$\sum F = F - F_1 - F_2 = 0$$

Same deflections

$$\delta_1 = \delta_2 = \delta$$

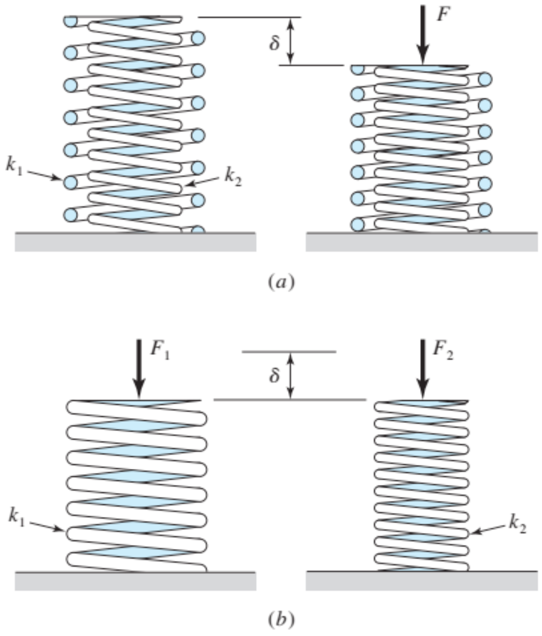
$$\frac{F_1}{k_1} = \frac{F_2}{k_2}$$

$$F - \frac{k_1}{k_2}F_2 - F_2 = 0 \quad \text{or} \quad F_2 = \frac{k_2 F}{k_1 + k_2}$$

$$F_1 = k_1 F/(k_1 + k_2)$$

$$k = F/\delta = k_1 + k_2.$$

the nested helical springs



This is a simple example. Common situations are usually more complicated.

So how to do it in general?

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Procedures 1 & 2

for general statically indeterminate problems.

Procedure 1

1. Choose the redundant reaction(s). There may be alternative choices (See Example 4–14).
2. Write the equations of static equilibrium for the remaining reactions in terms of the applied loads and the redundant reaction(s) of step 1.
3. Write the deflection equation(s) for the point(s) at the locations of the redundant reaction(s) of step 1 in terms of the applied loads and the redundant reaction(s) of step 1. Normally the deflection(s) is (are) zero. If a redundant reaction is a moment, the corresponding deflection equation is a rotational deflection equation.
4. The equations from steps 2 and 3 can now be solved to determine the reactions.

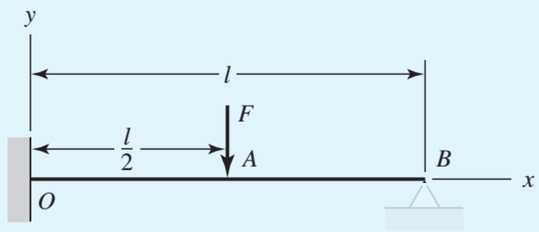
(Can be solved in any of the standard ways.)

Procedure 2

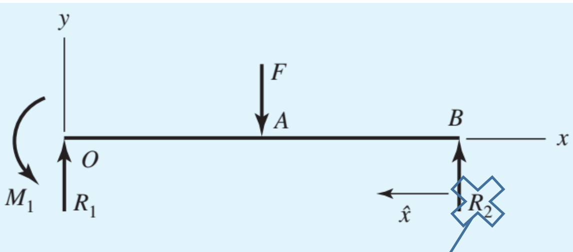
1. Write the equations of static equilibrium for the beam in terms of the applied loads and unknown restraint reactions.
2. Write the deflection equation for the beam in terms of the applied loads and unknown restraint reactions.
3. Apply boundary conditions to the deflection equation of step 2 consistent with the restraints.
4. Solve the equations from steps 1 and 3.

Example (*using superposition*):
Determine the reactions using procedure 1.

Beam 11 of Appendix Table A-9



(a)



(b)

Selected as Redundant Reaction

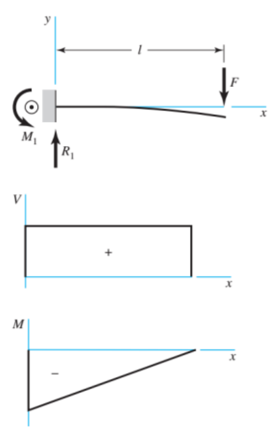
1 Choose R_2 at B to be the redundant reaction.

2 Using static equilibrium equations solve for R_1 and M_1 in terms of F and R_2. This results in

$$R_1 = F - R_2 \qquad M_1 = \frac{Fl}{2} - R_2 l \tag{1}$$

3 Write the deflection equation for point B in terms of F and R_2.

1 Cantilever—end load



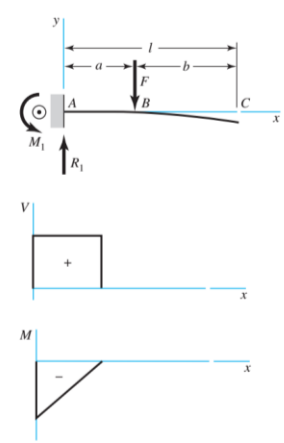
$R_1 = V = F \qquad M_1 = Fl$

$M = F(x - l)$

$y = \frac{Fx^2}{6EI}(x - 3l)$

$y_{max} = -\frac{Fl^3}{3EI}$

2 Cantilever—intermediate load



$R_1 = V = F \qquad M_1 = Fa$

$M_{AB} = F(x - a) \qquad M_{BC} = 0$

$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$

$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$

$y_{max} = \frac{Fa^2}{6EI}(a - 3l)$

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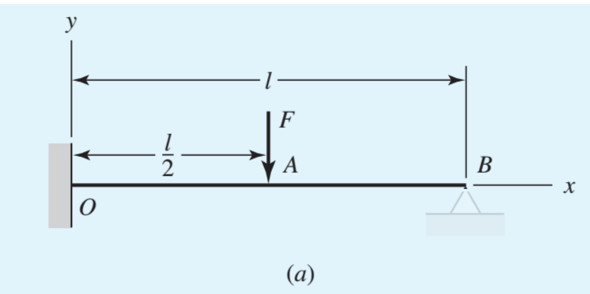
Example (*using superposition*): Determine the reactions using procedure 1.

Beam 11 of Appendix Table A-9

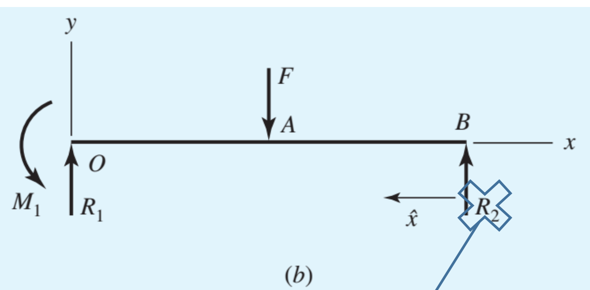


1 Choose R_2 at B to be the redundant reaction.

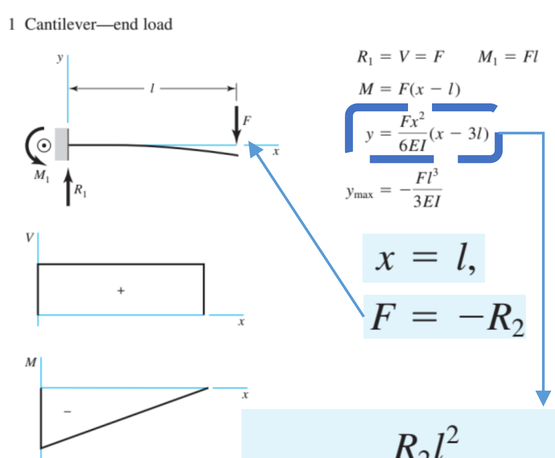
2 Using static equilibrium equations solve for R_1 and M_1 in terms of F and R_2. This results in

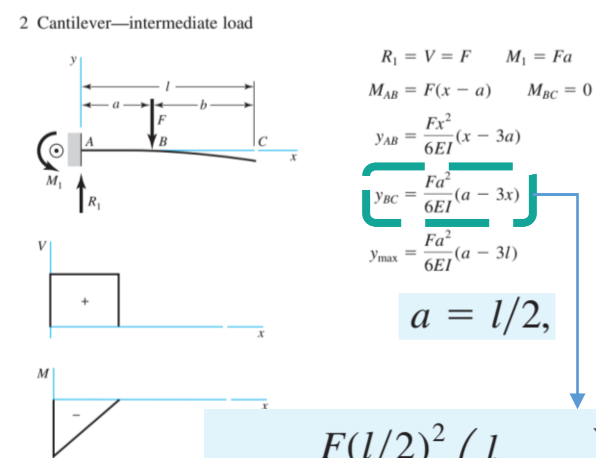
$$R_1 = F - R_2 \qquad M_1 = \frac{Fl}{2} - R_2 l \tag{1}$$

3 Write the deflection equation for point B in terms of F and R_2.



Selected as Redundant Reaction





$x = l,$

$F = -R_2$

$a = l/2,$

$$\delta_{B1} = -\frac{R_2 l^2}{6EI}(l - 3l)$$

$$\delta_{B2} = \frac{F(l/2)^2}{6EI}\left(\frac{l}{2} - 3l\right)$$

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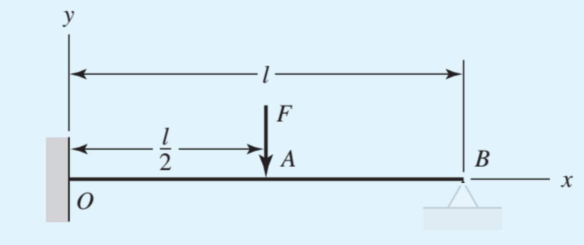
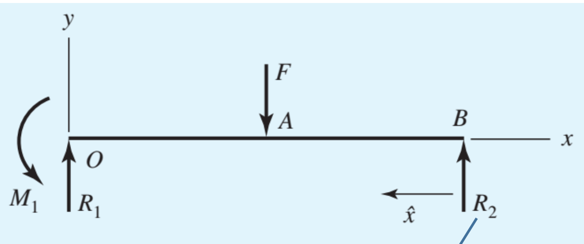
Example (*using superposition*):
Determine the reactions using procedure 1.

Beam 11 of Appendix Table A-9



(a)



(b)

Selected as Redundant Reaction

$$R_1 = F - R_2 \qquad M_1 = \frac{Fl}{2} - R_2 l \qquad (1)$$

$$\delta_{B1} = -\frac{R_2 l^2}{6EI}(l - 3l)$$

$$\delta_{B2} = \frac{F(l/2)^2}{6EI}\left(\frac{l}{2} - 3l\right)$$

$$\delta_B = 0$$

$$\delta_B = -\frac{R_2 l^2}{6EI}(l - 3l) + \frac{F(l/2)^2}{6EI}\left(\frac{l}{2} - 3l\right) = \frac{R_2 l^3}{3EI} - \frac{5Fl^3}{48EI} = 0 \qquad (2)$$

4 Equation (2) can be solved for R_2 directly. This yields

$$R_2 = \frac{5F}{16} \qquad (3)$$

Next, substituting R_2 into Eqs. (1) completes the solution, giving

$$R_1 = \frac{11F}{16} \qquad M_1 = \frac{3Fl}{16} \qquad (4)$$

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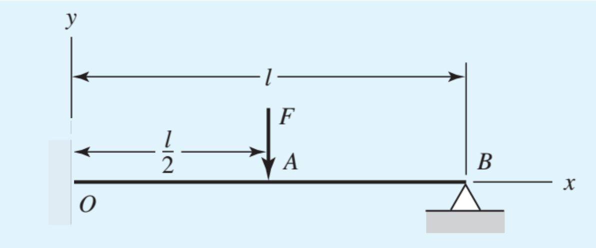
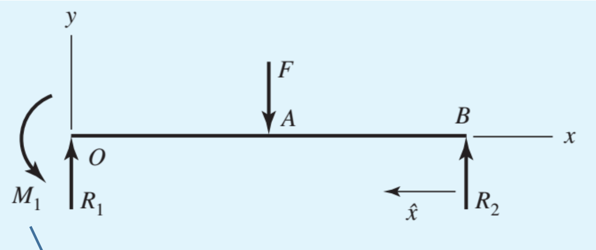
Example (*using Castigliano's Theorem*): Determine the reactions using procedure 1.

Beam 11 of Appendix Table A-9



(a)



(b)

Selected as Redundant Reaction

1 Choose M_1 at O to be the redundant reaction.

2 Using static equilibrium equations solve for R_1 and R_2 in terms of F and M_1. This results in

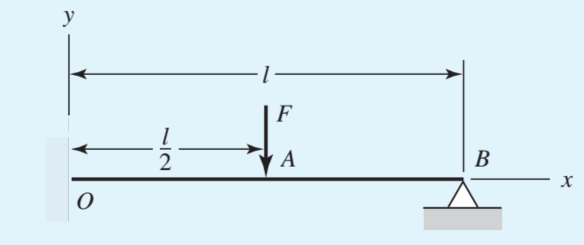
$$R_1 = \frac{F}{2} + \frac{M_1}{l} \qquad R_2 = \frac{F}{2} - \frac{M_1}{l} \tag{5}$$

3 Since M_1 is the redundant reaction at O, write the equation for the angular deflection at point O. From Castigliano's theorem this is

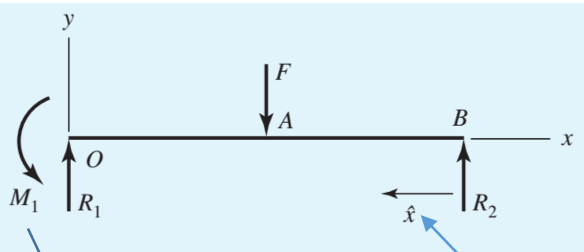
$$\theta_O = \frac{\partial U}{\partial M_1} \tag{6}$$

Example (*using Castigliano's Theorem*): Determine the reactions using procedure 1.

Beam 11 of Appendix Table A-9



(a)



(b)

Selected as Redundant Reaction

We can apply Eq. (4–31), using the variable x as shown in Fig. 4–16*b*. However, simpler terms can be found by using a variable $\hat{x}$ that starts at B and is positive to the left. With this and the expression for R_2 from Eq. (5) the moment equations are

$$M = \left(\frac{F}{2} - \frac{M_1}{l}\right)\hat{x} \qquad 0 \le \hat{x} \le \frac{l}{2} \tag{7}$$

$$M = \left(\frac{F}{2} - \frac{M_1}{l}\right)\hat{x} - F\left(\hat{x} - \frac{l}{2}\right) \qquad \frac{l}{2} \le \hat{x} \le l \tag{8}$$

For both equations

$$\frac{\partial M}{\partial M_1} = -\frac{\hat{x}}{l} \tag{9}$$

Substituting Eqs. (7) to (9) in Eq. (6), using the form of Eq. (4–31) where $F_i = M_1$, gives

$$\theta_O = \frac{\partial U}{\partial M_1} = \frac{1}{EI}\left\{\int_0^{l/2}\left(\frac{F}{2} - \frac{M_1}{l}\right)\hat{x}\left(-\frac{\hat{x}}{l}\right)d\hat{x} + \int_{l/2}^{l}\left[\left(\frac{F}{2} - \frac{M_1}{l}\right)\hat{x}\right.\right.$$

$$\left.\left.- F\left(\hat{x} - \frac{l}{2}\right)\right]\left(-\frac{\hat{x}}{l}\right)d\hat{x}\right\} = 0$$

$$M_1 = \frac{3Fl}{16} \tag{10}$$

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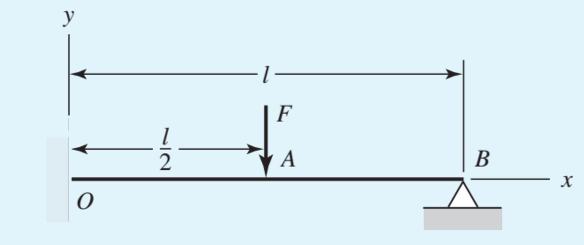
Example (*using Castigliano's Theorem*): Determine the reactions using procedure 1.

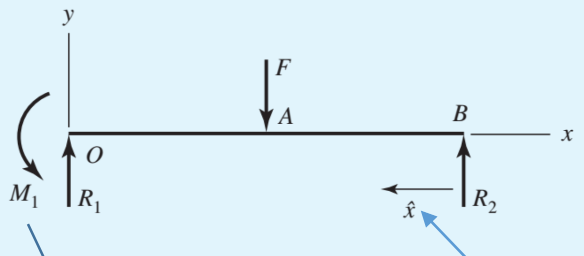
Beam 11 of Appendix Table A-9



(*a*)

$$M_1 = \frac{3Fl}{16} \qquad (10)$$

$$R_1 = \frac{F}{2} + \frac{M_1}{l} \qquad R_2 = \frac{F}{2} - \frac{M_1}{l} \qquad (5)$$



(*b*)

Selected as Redundant Reaction

4 Substituting Eq. (10) into (5) results in

$$R_1 = \frac{11F}{16} \qquad R_2 = \frac{5F}{16} \qquad (11)$$

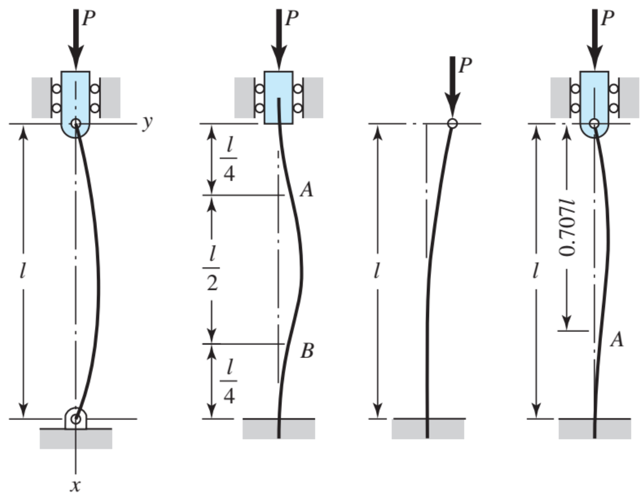
The same results are derived.

Global Instability—Buckling

Compressive loads/stresses within any long, thin structure can cause structural instabilities.

1. Long columns with central loading
2. Intermediate-length columns with central loading
3. Columns with eccentric loading
4. Struts or short columns with eccentric loading

The column becomes ***unstable*** when *P (still at a relatively low value)* reaches a specific value, causing bending to develop rapidly.



(*a*) $C = 1$ (*b*) $C = 4$ (*c*) $C = \frac{1}{4}$ (*d*) $C = 2$

Column End Conditions	End-Condition Constant *C*: Theoretical Value	Conservative Value	Recommended Value*
Fixed-free	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Rounded-rounded	1	1	1
Fixed-rounded	2	1	1.2
Fixed-fixed	4	1	1.2

*To be used only with liberal factors of safety when the column load is accurately known.

Euler column formula for Critical Load

$$P_{cr} = \frac{C\pi^2 EI}{l^2}$$

Critical Unit Load

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}$$

l/k is called the *slenderness ratio.*

Commonly used to classify columns

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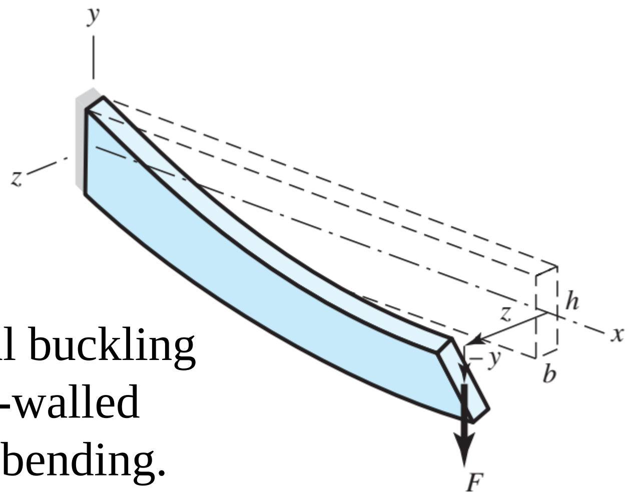
Elastic Stability

Be aware of the potential safety issues.



Torsional buckling of a thin-walled beam in bending.

- If the beam is long enough and the ratio of *b/h* is sufficiently small, there is a critical value of F for which the beam will collapse in a twisting mode as shown.
- This is due to the *compression* in the bottom fibers of the beam which cause the fibers to buckle sideways (z direction).

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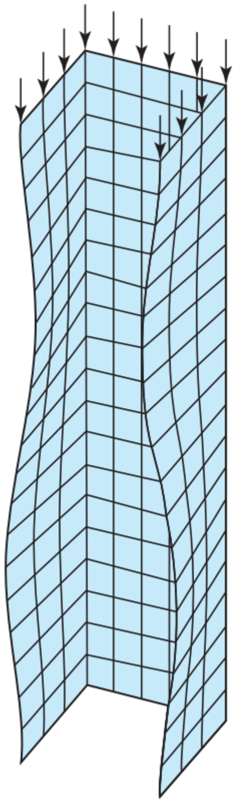
Finite-element representation may be necessary in certain unique cases, i.e. flange buckling of a channel in compression.



Outside the scope of this course.

Shock & Impact

Shock describes any suddenly applied force or disturbance.

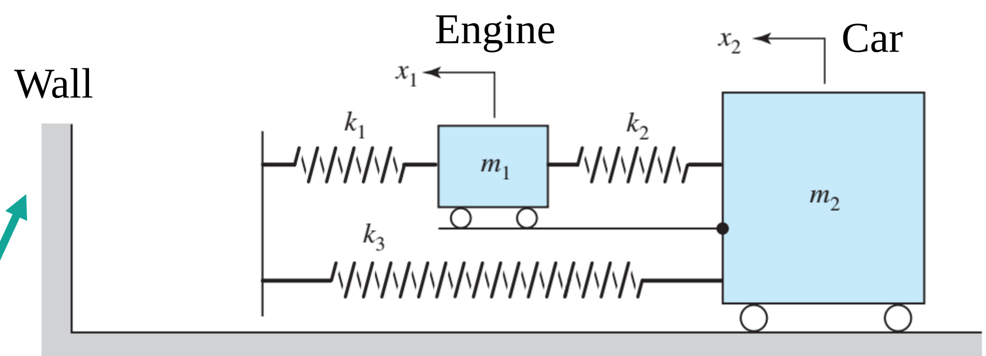
Impact refers to the collision of two masses with initial relative velocity.

- Desirable Impact
- Undesirable Impact

i.e. Stamping Machine

Gear Transmission

A highly simplified mathematical model of an automobile in collision with a rigid obstruction.



$$m\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + k_3x_2 - k_2(x_1 - x_2) = 0$$

Advanced courses in Mechanical Vibration is a good point of continuation.

Next class

- **Discussion for Group 1**: Mechanism Design
- Friday 0800-1000, Sep 20
- Room 202, 1 Lychee Park

- **Lab for Group 2**: Mechanism Design
- Friday 0800-1000, Sep 20
- Room 412, 5 Wisdom Valley

Thank you!

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