ME303 Introduction to Mechanical Design

Lecture 03
Kinematics & Load Analysis

Song Chaoyang

Assistant Professor

Department of Mechanical and Energy Engineering

songcy@sustech.edu.cn

SUSTech
Southern University
of Science and Technology

Agenda

Week 02, Wednesday, Sep 11, 2019

- Mechanism Basics
  - Kinematics | Degree of Freedom | Mechanism | Mobility
- Common 1-DOF Mechanisms
  - Fourbar Linkage | Fourbar Crank-Slider | Cam & Follower
- Analyzing Linkage Motion
- Equilibrium & Free-body Diagrams
- Load Analysis
  - Two-Dimensional | Three-Dimensional | Fourbar Linkage Case Study | Vibration Loading | Beam Loading

Kinematics

The study of motion without regard to forces.

- Degree of Freedom
  - *The number of coordinates needed to define its position in space*
    - One-DOF Mechanism: DOF = 1
    - Multi-DOF Mechanism: DOF > 1
    - Structure: DOF = 0
    - Preloaded Structure: DOF < 1

- Mechanisms: variants of a linkage
  - *A collection of **links** and **joints**, one of which is grounded, and all are interconnected in a way to provide a controlled output in response to one or more inputs.*

AncoraSIR.com

SUSTech Southern University of Science and Technology

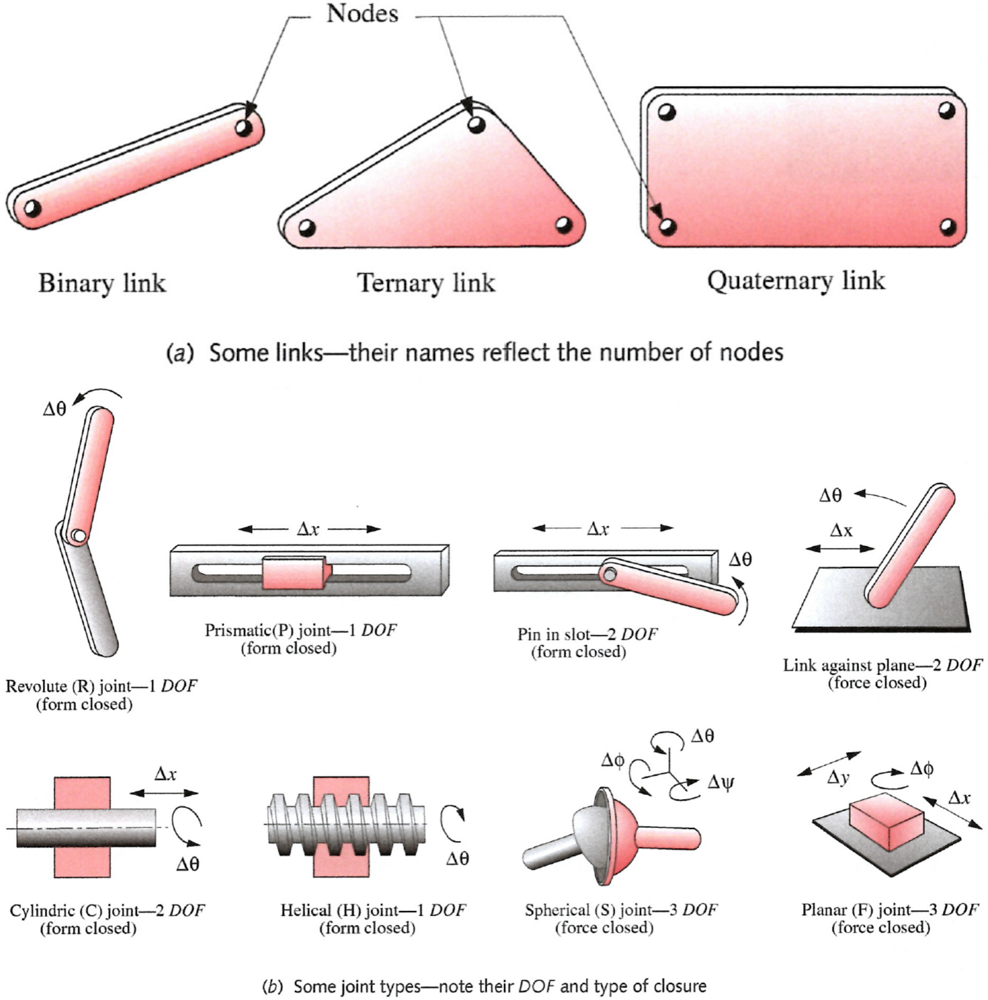
Links, Joints & Mobility

The construction of a linkage

- Link
  - A rigid body of any shape that has some number of attachment points called nodes, that allow multiple links to be connected by joints
- Joints
  - Characterized by their geometry, by the number of DOF they allow between the links they join, and by whether they are held together (closed) by a ***force*** or by their ***form*** (geometry)
- Kutzbach Equation

$$M = 3(L-1) - 2J_1 - J_2$$

  - M: Mobility (DOF)
  - L: the number of links
  - J_1: the number of one-DOF joints
  - J_2: the number of two-DOF joints
  - ***[Doesn't really work]***

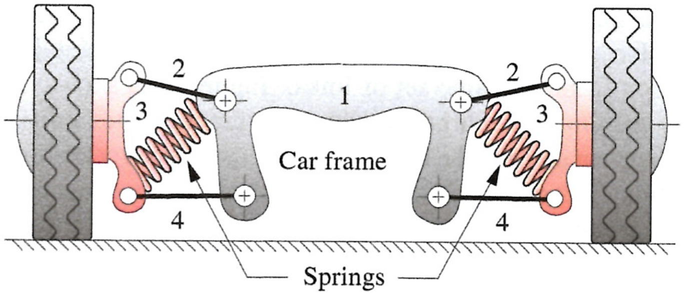


AncoraSIR.com

The Fourbar Linkage

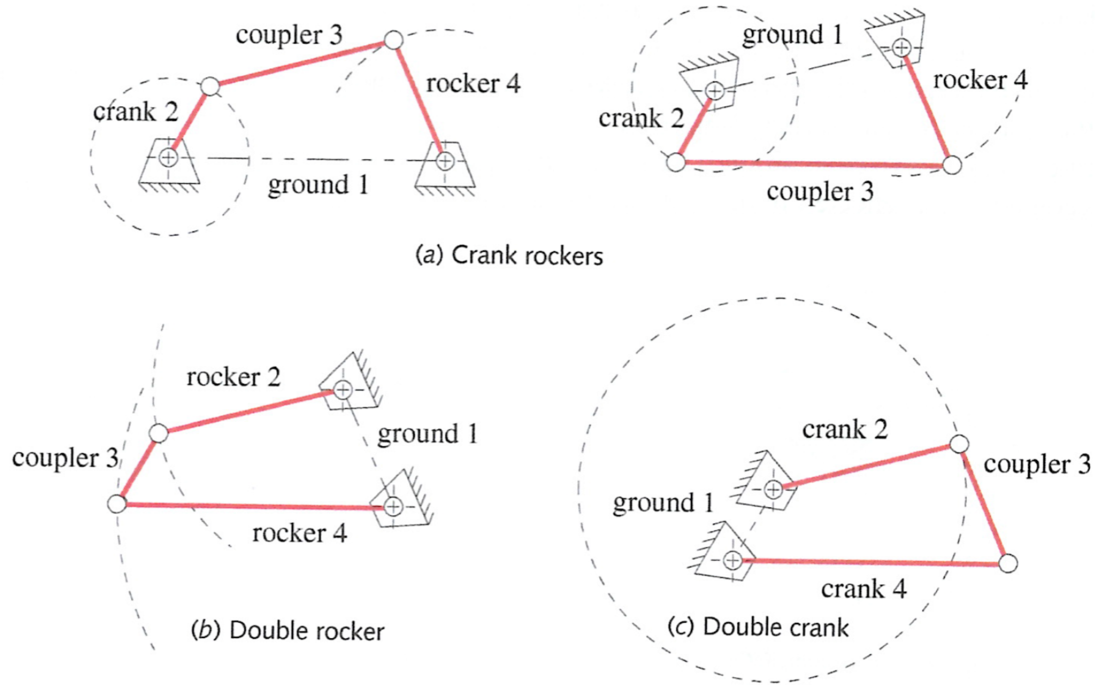
Four binary links connected by four pin joints.

- Grashof Criterion $S + L \leq P + Q$
  - S / L: The length of the shortest / longest link
  - P & Q: The lengths of the other two



- A Grashof Linkage
  - At least one link can revolve fully
  - It has “change-point” positions when equal [Singularity]

Fourbar linkage is simply the best linkage of all.

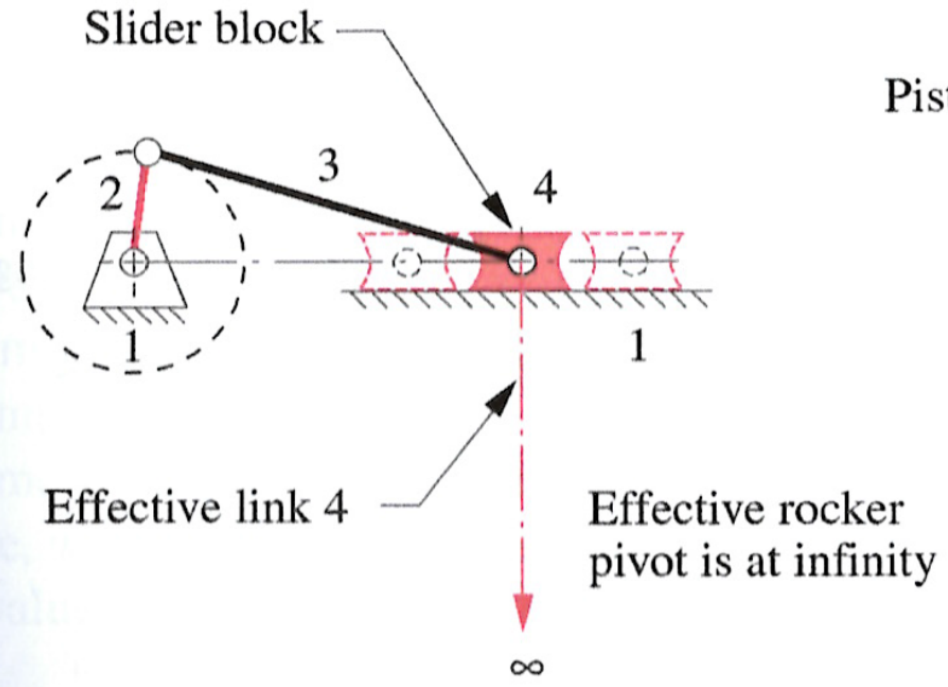


AncoraSIR.com

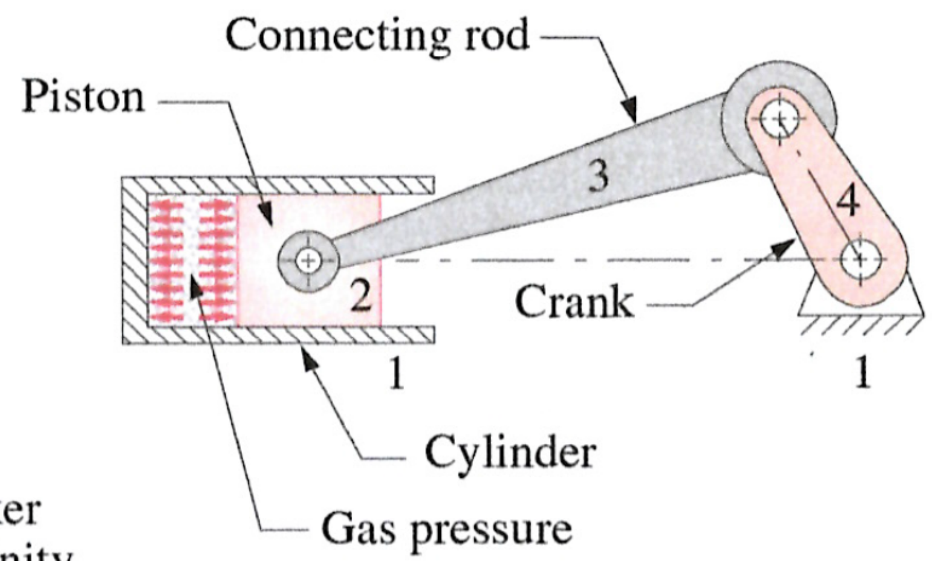
Fourbar Crank-Slider & Slider-Crank

Variants of the Fourbar Linkage

- *If a rocker of a fourbar crank rocker linkage is increased in length indefinitely, it effectively becomes infinite in length and the linkage is transformed into a fourbar crank-slider, which is similar for a fourbar slider-crank.*



(*a*) Crank-slider—crank drives slider



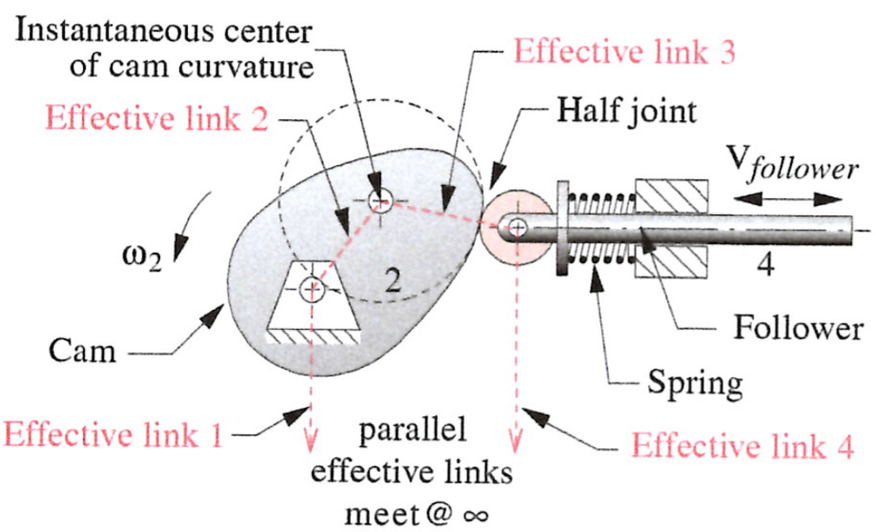
(*b*) Slider-crank—slider drives crank

Cam and Follower

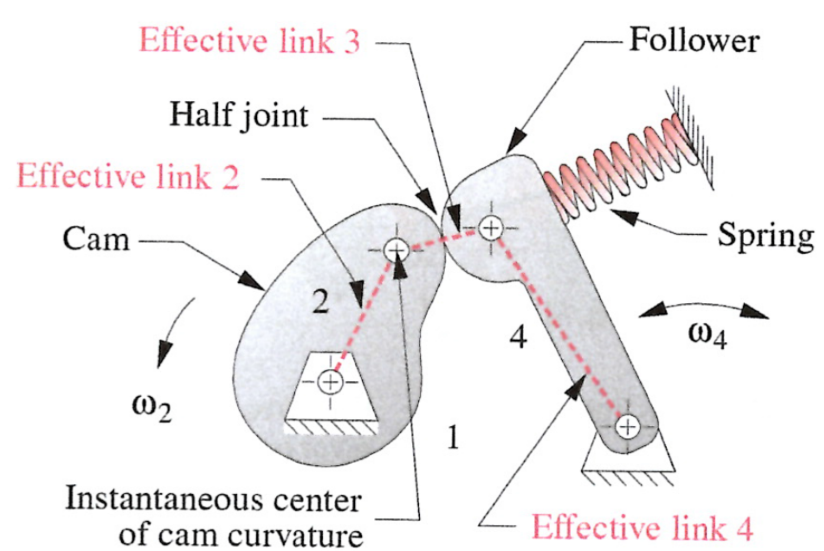
Another Variant of the Fourbar Linkage

- *The crank takes a contoured shape.*

- *Can be viewed as a fourbar crank-slider or slider-crank in which the crank and coupler are able to change their lengths as the cam rotates.*



(*a*) Cam with sliding follower—a variant of a crank-slider

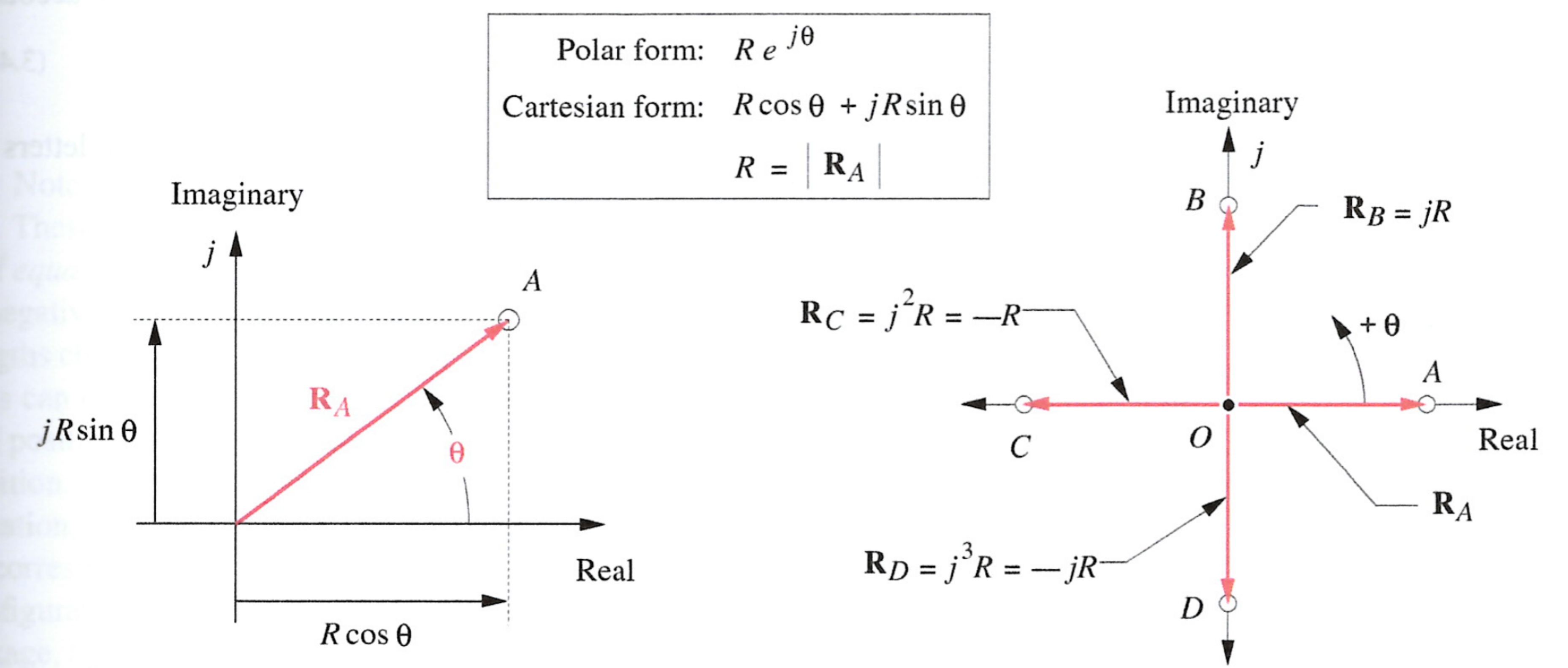


(*b*) Cam with oscillating follower—a variant of a crank-rocker

Analyzing Linkage Motion

Complex Motion = Translation (x, y) + Rotation (theta)

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$



(a) Complex number representation of a position vector

(b) Vector rotations in the complex plane

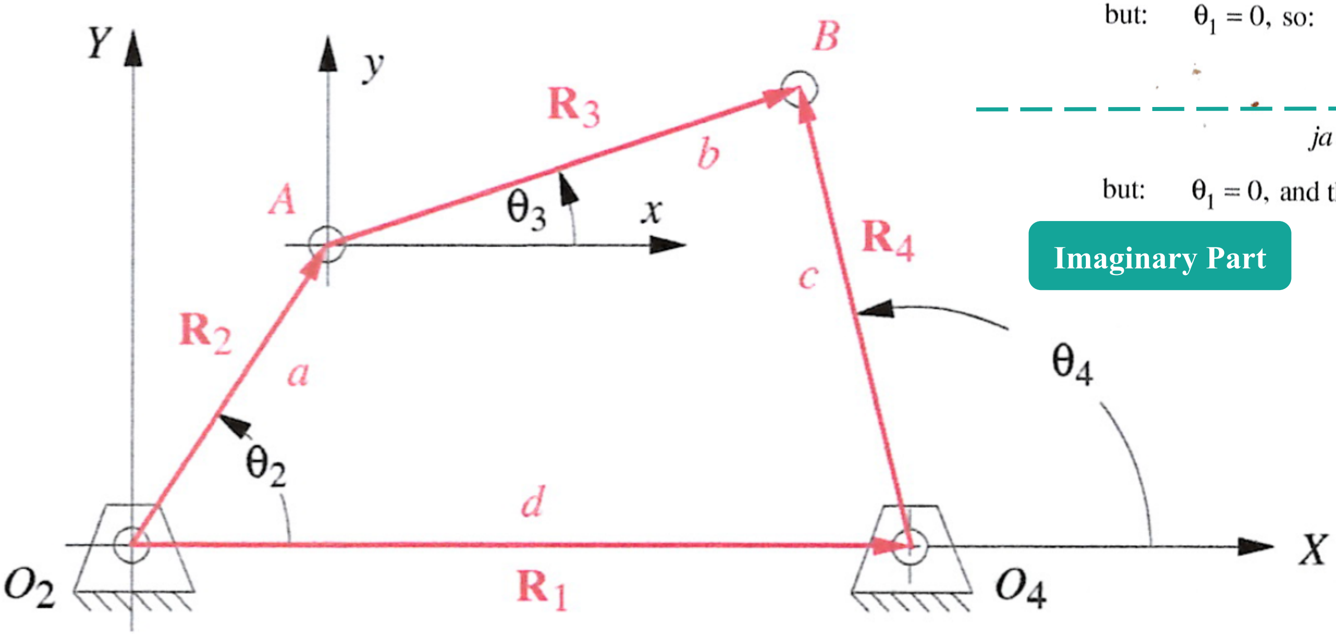
Analyzing the Fourbar Linkage

Kinematic Analysis: **Position** => Velocity => Acceleration

Vector Loop Equation

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \quad \Rightarrow \quad a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

$$a(\cos\theta_2 + j\sin\theta_2) + b(\cos\theta_3 + j\sin\theta_3) - c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0$$



Real Part

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so:

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$$

$$ja\sin\theta_2 + jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the j's divide out, so:

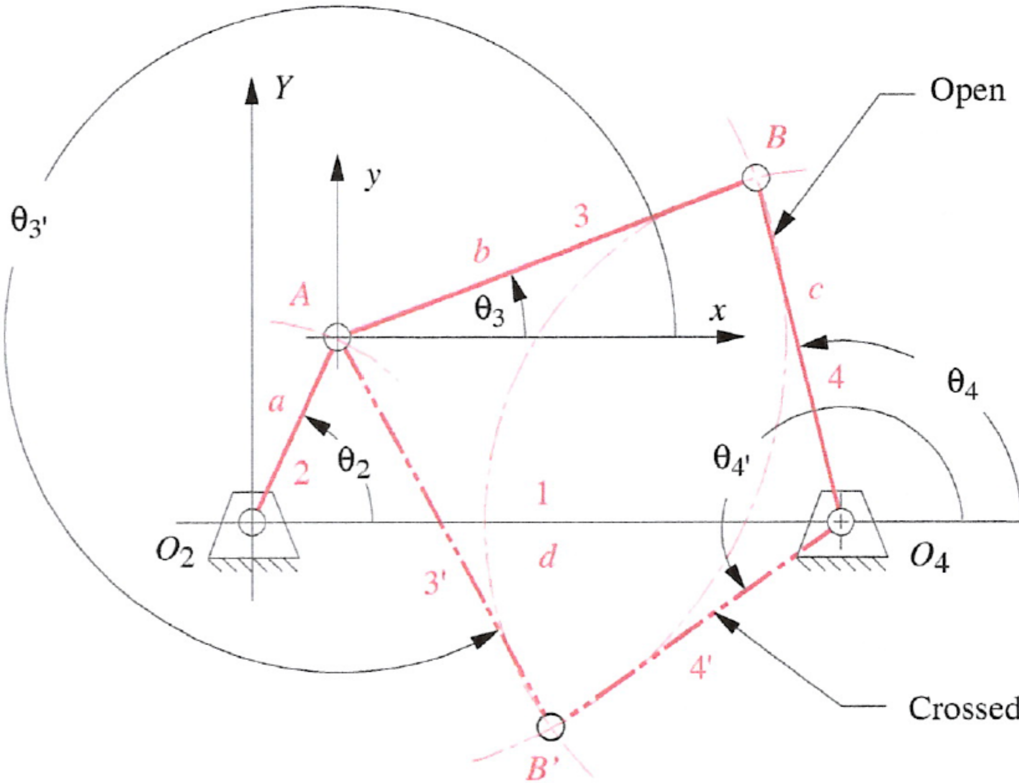
Imaginary Part

$$a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 = 0$$

SUSTech Southern University of Science and Technology

The Two Solutions to theta_3/4 of the Fourbar **Position** Equation

Crossed vs. Open solutions



$$\theta_{3_{1,2}} = 2\arctan\left(\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right)$$

where

$$D = \cos\theta_2 - K_1 + K_4\cos\theta_2 + K_5$$
$$E = -2\sin\theta_2$$
$$F = K_1 + (K_4 - 1)\cos\theta_2 + K_5$$

and

$$K_1 = \frac{d}{a} \qquad K_4 = \frac{d}{b} \qquad K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab}$$

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right)$$

where:

$$A = \cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3$$
$$B = -2\sin\theta_2$$
$$C = K_1 - (K_2 + 1)\cos\theta_2 + K_3$$

and:

$$K_1 = \frac{d}{a}, \qquad K_2 = \frac{d}{c}, \qquad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

AncoraSIR.com

SUSTech
Southern University of Science and Technology

Velocity Equation of the Fourbar Linkage

Kinematic Analysis: Position => **Velocity** => Acceleration

Vector Loop Equation

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \quad \Rightarrow \quad a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

Differentiate with respect to time to get velocity.

$$ja e^{j\theta_2} \frac{d\theta_2}{dt} + jb e^{j\theta_3} \frac{d\theta_3}{dt} - jc e^{j\theta_4} \frac{d\theta_4}{dt} = 0$$

but, $\frac{d\theta_2}{dt} = \omega_2; \quad \frac{d\theta_3}{dt} = \omega_3; \quad \frac{d\theta_4}{dt} = \omega_4$

so: $ja\omega_2 e^{j\theta_2} + jb\omega_3 e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0$

$$\mathbf{V}_A = ja\omega_2(\cos\theta_2 + j\sin\theta_2) = a\omega_2(-\sin\theta_2 + j\cos\theta_2)$$

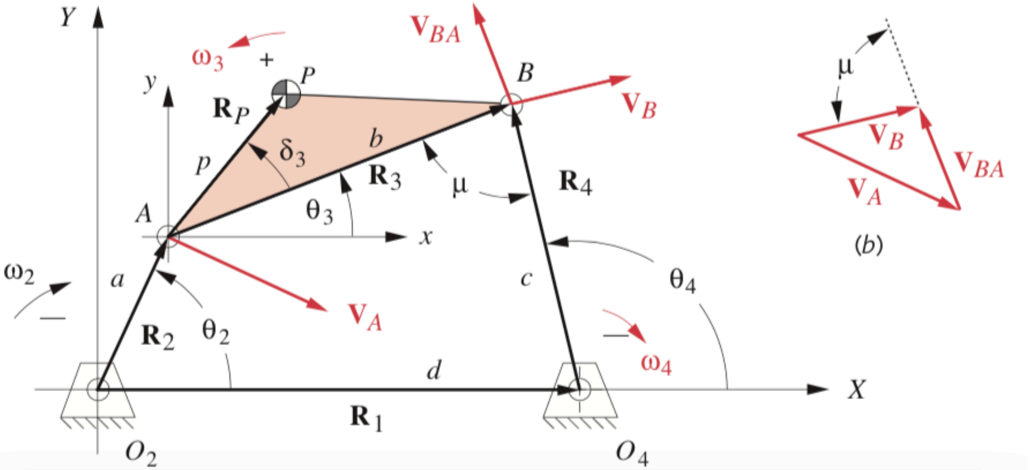
$$\mathbf{V}_{BA} = jb\omega_3(\cos\theta_3 + j\sin\theta_3) = b\omega_3(-\sin\theta_3 + j\cos\theta_3)$$

$$\mathbf{V}_B = jc\omega_4(\cos\theta_4 + j\sin\theta_4) = c\omega_4(-\sin\theta_4 + j\cos\theta_4)$$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$\omega_3 = \frac{a\omega_2}{b}\frac{\sin(\theta_4 - \theta_2)}{\sin(\theta_3 - \theta_4)}$$

$$\omega_4 = \frac{a\omega_2}{c}\frac{\sin(\theta_2 - \theta_3)}{\sin(\theta_4 - \theta_3)}$$



AncoraSIR.com

Acceleration Equation of the Fourbar Linkage

Kinematic Analysis: Position => Velocity => **Acceleration**

Vector Loop Equation $\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0$ → $a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$

*Differentiate with respect to time to get **velocity**.*

$$ja\omega_2 e^{j\theta_2} + jb\omega_3 e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0$$

*Differentiate with respect to time to get **acceleration**.*

$$\left(j^2 a\omega_2^2 e^{j\theta_2} + ja\alpha_2 e^{j\theta_2}\right) + \left(j^2 b\omega_3^2 e^{j\theta_3} + jb\alpha_3 e^{j\theta_3}\right) - \left(j^2 c\omega_4^2 e^{j\theta_4} + jc\alpha_4 e^{j\theta_4}\right) = 0$$

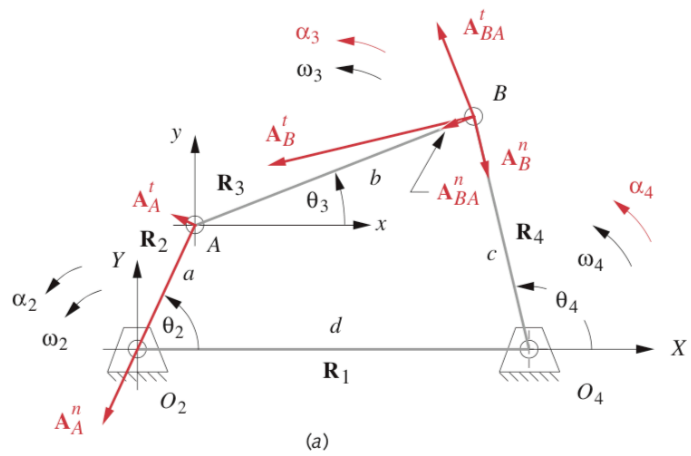
Rewrite for physical meaning.

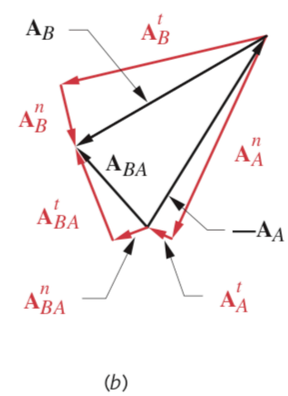
$$\mathbf{A}_A + \mathbf{A}_{BA} - \mathbf{A}_B = 0$$

$$\mathbf{A}_A = \left(\mathbf{A}_A^t + \mathbf{A}_A^n\right) = \left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right)$$

$$\mathbf{A}_{BA} = \left(\mathbf{A}_{BA}^t + \mathbf{A}_{BA}^n\right) = \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right)$$

$$\mathbf{A}_B = \left(\mathbf{A}_B^t + \mathbf{A}_B^n\right) = \left(c\alpha_4 je^{j\theta_4} - c\omega_4^2 e^{j\theta_4}\right)$$



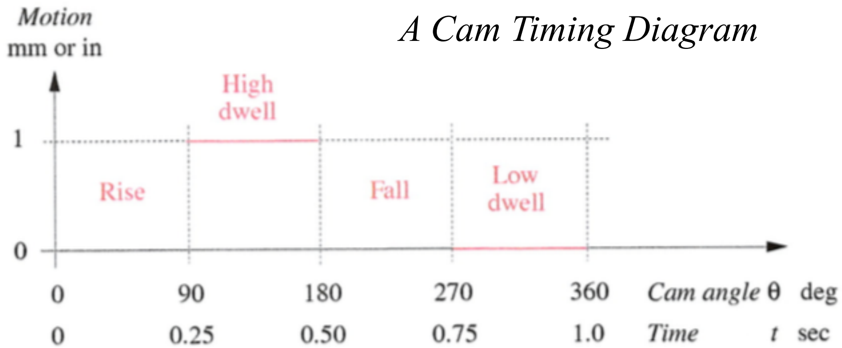


Cam Design & Analysis

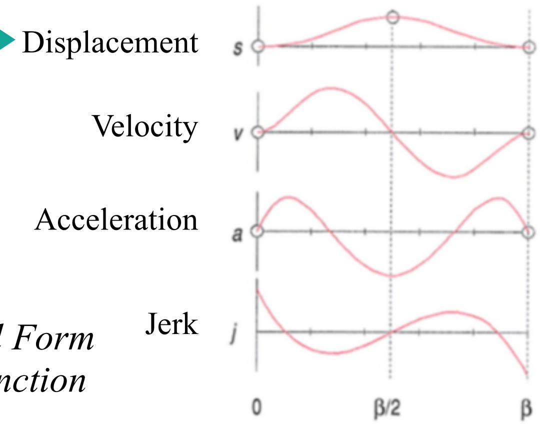
Cam-Follower Linkage

- Dwell
  - The cessation of follower motion while the cam rotation continues for a portion of the cycle

A Cam Timing Diagram



svaj Diagram



$$s = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \cdots + C_n x^n$$

The crank takes a contoured shape.

General Polynomial Form of Displacement Function

- Fundamental Law of Cam Design
  - *The motion function must be piecewise continuous over the entire cam through the second derivative of displacement.*
  - ***No discontinuities in the position, velocity, or acceleration functions over the full cycle of the cam.***

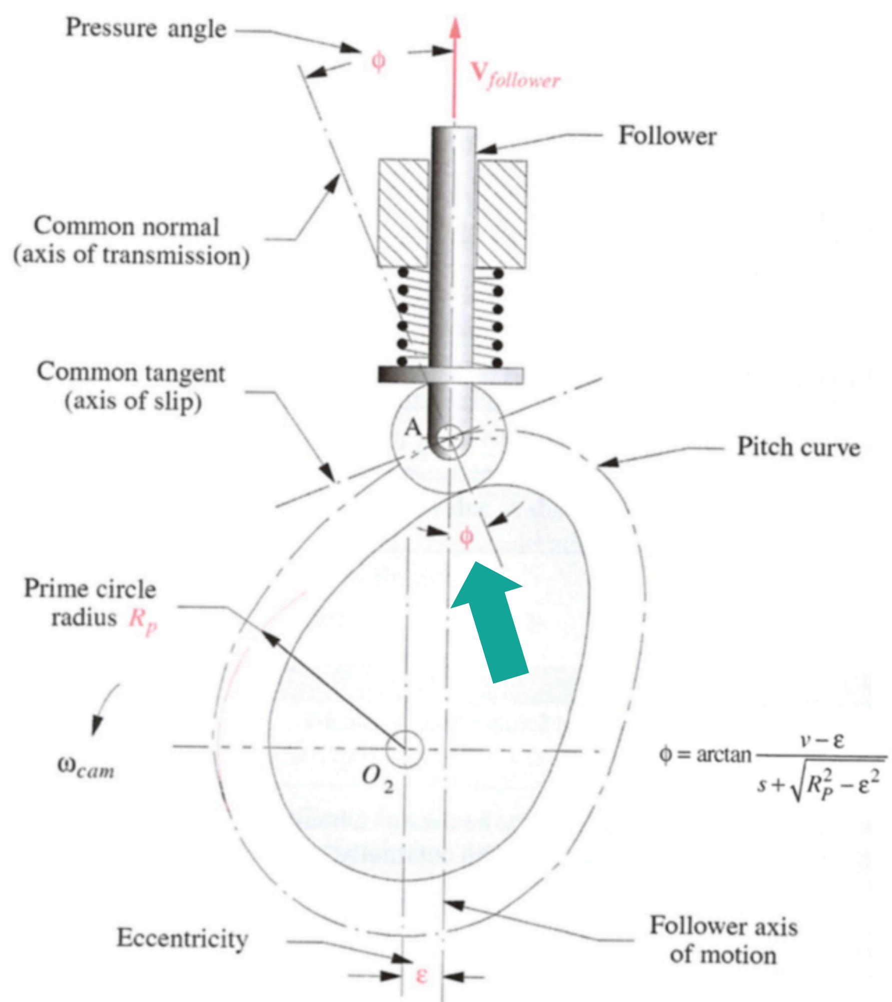
AncoraSIR.com

SUSTech
Southern University of Science and Technology

Pressure Angle

Determines what percent of the force goes into motion
and what percent is trying to jam the follower sideways in its guide.

- Definition
  - The angle between the velocity of the follower and the common normal at the contact point between cam and follower.
- Rule of thumb
  - Keep the maximum pressure angle below about 30 degree for a translating follower.

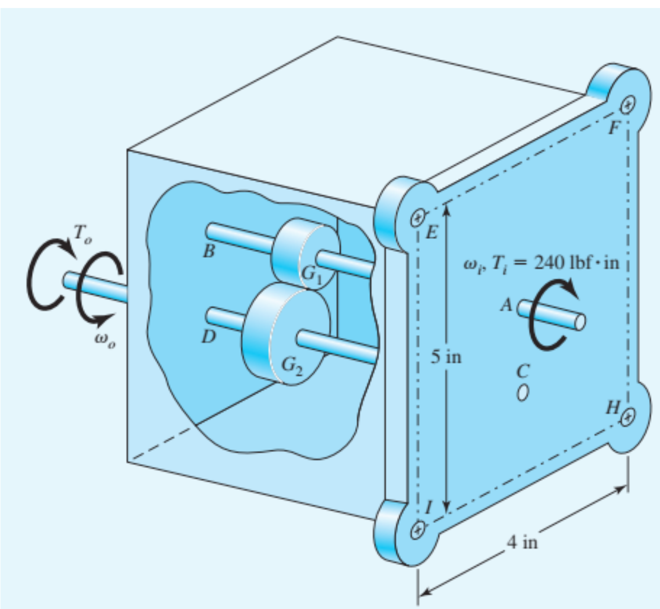


AncoraSIR.com

Equilibrium & Free-body Diagrams

One must understand the nature of forces before attempting to perform an extensive stress or deflection analysis of a mechanical component.

- **Equilibrium**: *A System with Zero Acceleration.*
- **Free-body Diagrams**: *A means of breaking a complicated problem into manageable segments, analyzing these simple problems, and then, usually, putting the information together again.*

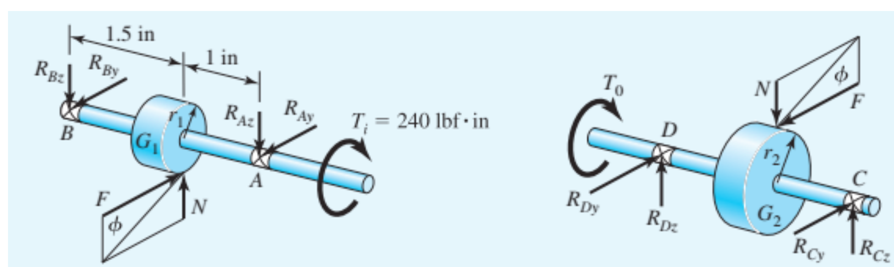


$$\sum \mathbf{F} = 0$$

$$\sum \mathbf{M} = 0$$

$$\sum M_x = F(0.75) - 240 = 0$$
$$F = 320 \text{ lbf}$$

$$R_{Ay} = 192 \text{ lbf},\ R_{Az} = 69.9 \text{ lbf},$$
$$R_{By} = 128 \text{ lbf},\ R_{Bz} = 46.6 \text{ lbf},$$
$$R_{Cy} = 192 \text{ lbf},\ R_{Cz} = 69.9 \text{ lbf},$$
$$R_{Dy} = 128 \text{ lbf},\ R_{Dz} = 46.6 \text{ lbf},$$
$$T_o = 480 \text{ lbf} \cdot \text{in}.$$



(*c*) Input shaft (*d*) Output shaft

AncoraSIR.com

Load Analysis

Three-Dimensional & Two-Dimensional Analysis

- Newton's **First** Law *(Governing Assumption)*
  - *A body at rest tends to remain at rest and a body in motion at constant velocity will tend to maintain that velocity unless acted upon by an external force.*
- Newton's **Second** Law *(Calculated Guess)*
  - *The time rate of change of momentum of a body is equal to the magnitude of the applied force and acts in the direction of the force.*
- Newton's **Third** Law *(Interactive Analysis)*
  - *When two particles interact, a pair of equal and opposite reaction forces will exist at their contact point. This force pair will have the same magnitude and act along the same direction line, but have opposite sense.*

$$\sum \mathbf{F} = m\mathbf{a} \qquad \sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

$$\sum \mathbf{M}_G = \dot{\mathbf{H}}_G \qquad \mathbf{H}_G = I_x\omega_x\hat{\mathbf{i}} + I_y\omega_y\hat{\mathbf{j}} + I_z\omega_z\hat{\mathbf{k}}$$

I_x, I_y, and I_z are the principal centroidal mass moments of inertia

These 6 equations can be written for each rigid body in a 3-D system

$$\sum M_x = I_x\alpha_x - (I_y - I_z)\omega_y\omega_z$$
$$\sum M_y = I_y\alpha_y - (I_z - I_x)\omega_z\omega_x$$
$$\sum M_z = I_z\alpha_z - (I_x - I_y)\omega_x\omega_y$$

Moments   *Angular Accelerations*

Needs to be solved simultaneously for the forces and moments

Total # of Load Eqs. to solve for a system = *# of rigid bodies the system* × *(3 Force Eqs. + 3 Moment Eqs.)* + *Reaction Eqs.*

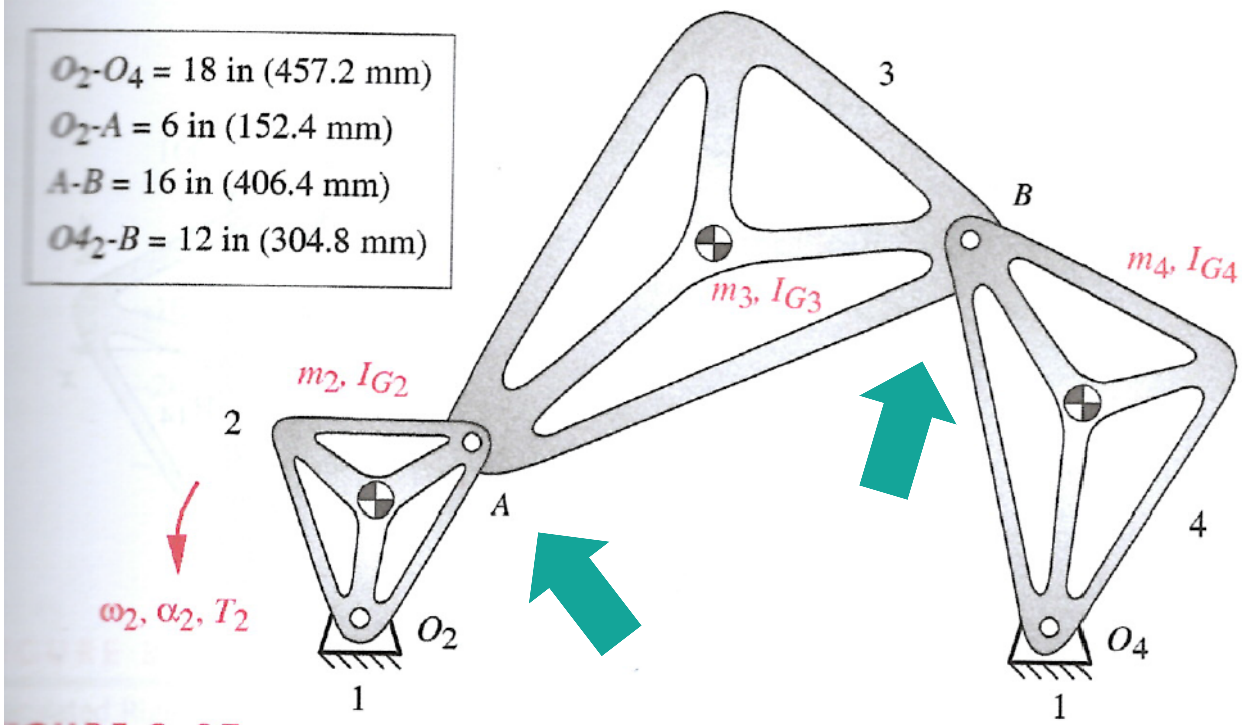
AncoraSIR.com

SUSTech Southern University of Science and Technology

Fourbar Linkage Loading Analysis

A Case Study

- [**Problem**] Determine the theoretical rigid body **forces** acting in two dimensions on the fourbar linkage.
- [**Given**] The linkage geometry, masses, and mass moments of inertia are known and the linkage is driven at up to 120 rpm by a speed-controlled electric motor.



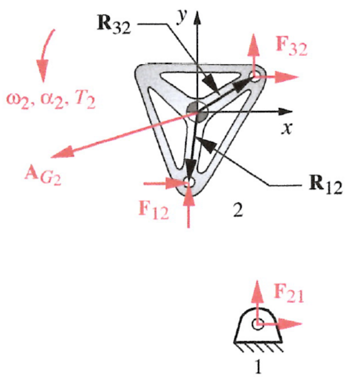
AncoraSIR.com

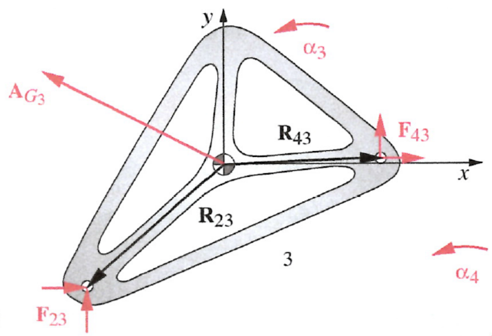
Free-Body Diagrams

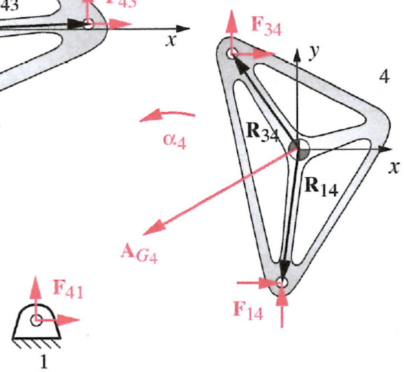
Elements in a Fourbar Linkage

Given and Assumed Data

Variable	Value	Unit
θ_2	30.00	deg
ω_2	120.00	rpm
$mass_2$	0.525	kg
$mass_3$	1.050	kg
$mass_4$	1.050	kg
I_{cg2}	0.057	kg-m^2
I_{cg3}	0.011	kg-m^2
I_{cg4}	0.455	kg-m^2
R_{12x}	–46.9	mm
R_{12y}	–71.3	mm
R_{32x}	85.1	mm
R_{32y}	4.9	mm
R_{23x}	–150.7	mm
R_{23y}	–177.6	mm
R_{43x}	185.5	mm
R_{43y}	50.8	mm
R_{14x}	–21.5	mm
R_{14y}	–100.6	mm
R_{34x}	–10.6	mm
R_{34y}	204.0	mm







$$F_{32x} = -F_{23x}$$
$$F_{32y} = -F_{23y}$$
$$F_{34x} = -F_{43x}$$
$$F_{34y} = -F_{43y}$$

For link 2:

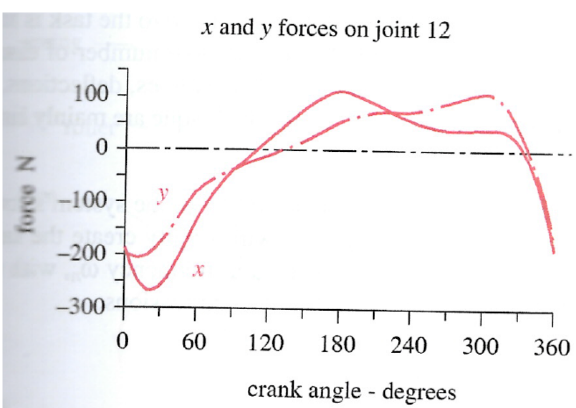
$$\sum F_x = F_{12x} + F_{32x} = m_2 a_{G2_x}$$
$$\sum F_y = F_{12y} + F_{32y} = m_2 a_{G2_y}$$
$$\sum M_z = T_2 + \left(R_{12x}F_{12y} - R_{12y}F_{12x}\right) + \left(R_{32x}F_{32y} - R_{32y}F_{32x}\right) = I_{G2}\alpha_2$$

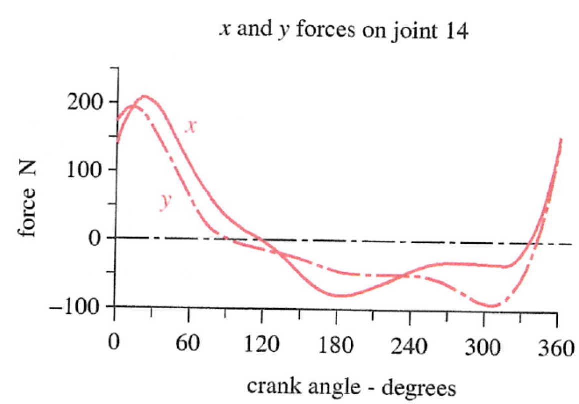
For link 3:

$$\sum F_x = F_{23x} + F_{43x} = m_3 a_{G3_x}$$
$$\sum F_y = F_{23y} + F_{43y} = m_3 a_{G3_y}$$
$$\sum M_z = \left(R_{23x}F_{23y} - R_{23y}F_{23x}\right) + \left(R_{43x}F_{43y} - R_{43y}F_{43x}\right) = I_{G3}\alpha_3$$

For link 4:

$$\sum F_x = F_{14x} + F_{34x} = m_4 a_{G4_x}$$
$$\sum F_y = F_{14y} + F_{34y} = m_4 a_{G4_y}$$
$$\sum M_z = \left(R_{14x}F_{14y} - R_{14y}F_{14x}\right) + \left(R_{34x}F_{34y} - R_{34y}F_{34x}\right) = I_{G4}\alpha_4$$





AncoraSIR.com

Vibration Loading

If the elements in the system were infinitely stiff, then vibrations would be eliminated.

- All real elements of any material have elasticity
  - Thus act as spring when subjected to forces
    - Causing deflection to produce additional forces to be generated from the inertial forces associated with the vibratory movement of elements;
    - If clearances allow contact of mating parts, may generate impact (shock) loads during their vibrations.
- How to predict?
  - Modern finite element (FEA) or boundary element (BEA) analysis techniques are good ways to model and calculate
    - Break up the assembly into a large number of discrete elements
    - Limited by time and the computing resources available
  - Field or Scaled experimentations
- How to eliminate?
  - Better Design or Better Design Engineers

AncoraSIR.com

Natural Frequency & Dynamic Forces

Model Vibration Loadings

- A system's lowest frequency (*a calculated estimation*)
  - Usually creates the largest magnitude of vibration

undamped fundamental natural frequency ω_n: $\omega_n = \sqrt{\frac{k}{m}}$

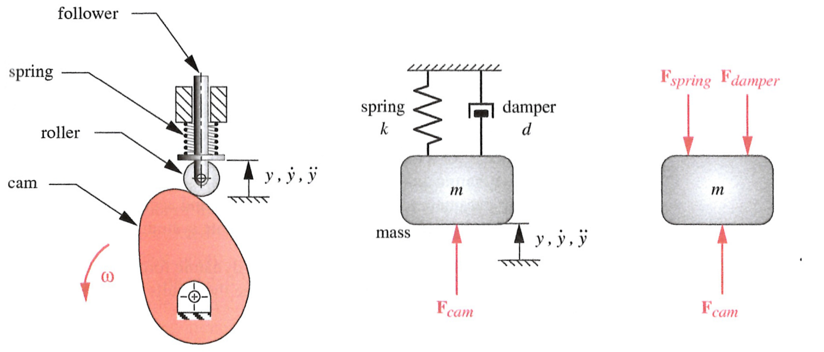
$f_n = \frac{1}{2\pi}\omega_n$

$$\sum F_y = ma = m\ddot{y}$$

$$F_{cam} - F_{spring} - F_{damper} = m\ddot{y}$$

$$F_{cam} = m\ddot{y} + d\dot{y} + ky$$

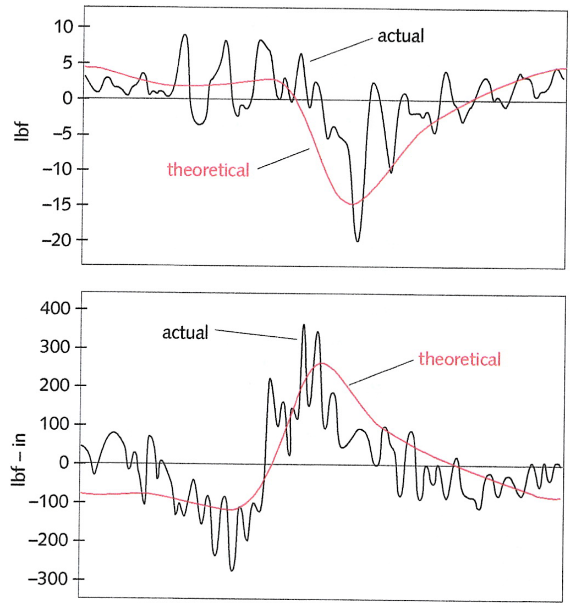
- Metrics
  - Spring Constant k $\quad k = \frac{F}{y}$
  - Damping d $\quad d = \frac{F}{\dot{y}}$



(*a*) Actual system (*b*) Lumped model (*c*) Free-body diagram

(*a*) Theoretical and actual dynamic force in *x* direction at crank pivot

(*b*) Theoretical and actual dynamic torque at crank pivot



AncoraSIR.com

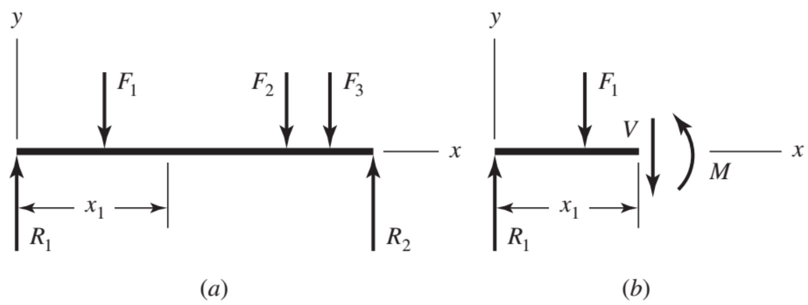
SUSTech Southern University of Science and Technology

Beam Loading

Shear Force & Bending Moments

Figure 3–2

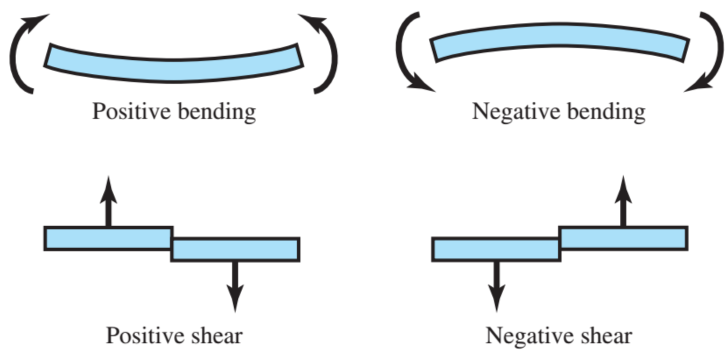
Free-body diagram of simply-supported beam with V and M shown in positive directions (established by the conventions shown in Fig. 3–3).



If the beam is cut at some section located at $x = x_1$ and the left-hand portion is removed as a free body, an internal shear force V and bending moment M must act on the cut surface to ensure equilibrium.

Figure 3–3

Sign conventions for bending and shear.



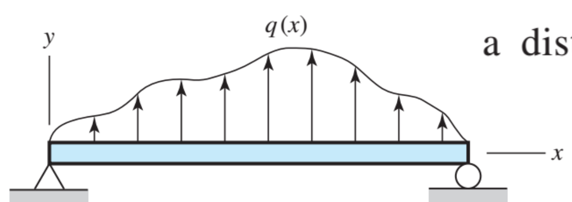
$$V = \frac{dM}{dx}$$

$$\int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q\, dx$$

$$\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V\, dx$$

Figure 3–4

Distributed load on beam.



a distributed load $q(x)$

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q$$

Advanced Textbooks are recommended to further study this topic in depth.

SUSTech
Southern University of Science and Technology

Next class

- **Lecture Topic**: Deflection & Stiffness
- Wednesday 1400-1600, Sep 18
- Room 206, 2 Lychee Park

Thank you!

Prof. Song Chaoyang (songcy@sustech.edu.cn)

- Xiao Xiaochuan (xiaoxc@sustech.edu.cn)

AncoraSIR.com

SUSTech
Southern University of Science and Technology